

THE FIFTH MOMENT OF MODULAR L -FUNCTIONS

EREN MEHMET KIRAL, MATTHEW P. YOUNG

ABSTRACT. We prove a sharp bound on the fifth moment of modular L -functions of fixed small weight, and large prime level.

1. INTRODUCTION

Let q be a prime and $\kappa \in \{4, 6, 8, 10, 14\}$. Let $H_\kappa(q)$ be the set of weight κ Hecke eigenforms on $\Gamma_0(q)$. For any $f \in H_\kappa(q)$ (note that any such f is automatically a newform), let $\lambda_f(n)$ denote its n^{th} Hecke eigenvalue. Our main result is the following theorem:

Theorem 1.1. *We have*

$$(1.1) \quad \sum_{f \in H_\kappa(q)} L(1/2, f)^5 \ll q^{1+\theta+\varepsilon},$$

as $q \rightarrow \infty$ among primes. Here θ is the best-known progress towards the Ramanujan-Petersson conjecture.

The currently best-known value $\theta = 7/64$ is given by the work of Kim and Sarnak [22]. The central value $L(1/2, f)$ is nonnegative by [23, Corollary 2] and [35], so upon dropping all but one term, we deduce:

Corollary 1.2. *For any $\varepsilon > 0$, we have*

$$(1.2) \quad L(1/2, f) \ll_\varepsilon q^{\frac{1+\theta}{5}+\varepsilon}.$$

Previously, Duke, Friendlander, and Iwaniec [10] bounded the amplified fourth moment in this family, and Kowalski, Michel, and VanderKam [24] asymptotically evaluated a mollified fourth moment. Recently, Balkanova and Frolenkov [2] improved the error term in these fourth moment problems, and thereby deduced the so-far best-known subconvexity result of $L(1/2, f) \ll q^{\frac{27-30\theta}{112-128\theta}}$. Corollary 1.2 improves this further. Our method of proof takes a different course from these works, and we never solve a shifted convolution problem.

This paper has some common features with the cubic moment studied by Conrey and Iwaniec [7]. Their work also bounds a moment that is one larger than what one may consider the “barrier” to subconvexity. That is, for the family of L -functions they consider, an upper bound on the second moment that is Lindelöf-on-average leads back precisely to the convexity bound on an individual L -value. Similarly, the fourth moment is the “barrier” in the family of Theorem 1.1. In a sense, going one full moment beyond the barrier is a way of amplifying with the L -function itself. As far as the authors are aware, prior to Theorem 1.1, the only

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known instances of a sharp upper bound on a moment that is one larger than the barrier moment are the cubic moment and its generalizations [7] [17] [31] [36] [32].

In Section 2, we give a simplified sketch of the argument. The main overall difficulty in the problem is that we require a significant amount of cancellation in multivariable sums with divisor functions and Kloosterman sums. The main thrust of the argument is to apply summation formulas that shorten the lengths of summation, eventually obtaining a sum of Kloosterman sums. To this, we apply the Bruggeman-Kuznetsov formula, which leads to a fourth moment of Hecke-Maass L -functions twisted by $\lambda_j(q)$, with an additional average over the level. This is another incarnation of a Kuznetsov/Motohashi-type formula where a moment problem in one family of L -functions is related to another moment in a “dual” family (see [26, Section 1.1.3]). Along the way, we encounter many “fake” main terms, which turn out to be surprisingly difficult to estimate. A straightforward bound on these would only lead to $O(q^{5/4+\varepsilon})$ in Theorem 1.1, which would be trivial. We expect that all the “fake” main terms calculated in this paper should essentially cancel, but doing so is a daunting prospect. Instead, we show that with an appropriate choice of weight functions in the approximate functional equations, then all the fake main terms are bounded consistently with Theorem 1.1. The amplified/mollified fourth moment (cf. [10] [24]) also required a difficult analysis of the main terms, which arose from solving the shifted convolution problem. As such, it is not clear how to compare the main term calculations here with [10] [24]. The article [3, Section 1.2] has a more through discussion of the main term analysis with the shifted convolution approach.

One of the practical difficulties in applying the Bruggeman-Kuznetsov formula in applications is that one needs to recognize the particular shape of sum of Kloosterman sums one encounters (with coprimality and congruence conditions, etc.) as one associated to a group Γ , pair of cusps $\mathfrak{a}, \mathfrak{b}$, and nebentypus χ . To this end, in Section 4 we have identified all the Kloosterman sums belonging to the congruence subgroup $\Gamma_0(N)$ and at general Atkin-Lehner cusps (i.e., those cusps equivalent to ∞ under an Atkin-Lehner involution) with general Dirichlet characters. It turns out that there is a “correct” choice of scaling matrix to use when computing the Fourier coefficients and Kloosterman sums, a choice that ensures the multiplicativity of Fourier coefficients at the Atkin-Lehner cusps.

Another practical difficulty is estimating oscillatory integral transforms with weight functions depending on multiple variables, with numerous parameters. It is desirable to perform stationary phase analysis on a single variable at a time, yet still keep track of the behavior of the remaining variables in a succinct way. In Section 5 we have codified some properties of a family of weight functions that allows us to do this efficiently. The key stationary phase result, which is a modest generalization of work in [4], is stated as Lemma 5.5 below (this result also appears in [32] which was prepared in tandem with this document).

In the spectral analysis of a sum of Kloosterman sums, it is necessary to treat the Maass forms, holomorphic forms, and continuous spectrum. In our situation, the Maass forms and holomorphic forms are rather similar, and lead to a twisted fourth moment of GL_2 cuspidal L -functions. The continuous spectrum is similar in many respects, but a key difference is that the “dual” family of L -functions is essentially a sum of a product of eight Dirichlet L -functions at shifted arguments. One naturally wishes to treat the continuous spectrum on the same footing as the discrete spectrum, which requires shifting some contour integrals past the poles of the Dirichlet L -functions (which occur only when the character is principal). There is potentially a large loss in savings from these poles on the 1-line compared to the

contribution from the $1/2$ -line. Luckily, it turns out that there is some extra savings in the residues of the Dirichlet series (essentially, from considering only the principal characters) that balances against this loss. This savings ultimately arises from a careful calculation of the Fourier expansion of the Eisenstein series on $\Gamma_0(N)$ with arbitrary N , attached to an arbitrary cusp, expanded around any Atkin-Lehner cusp.

An astute reader may note that $\kappa = 2$ is not covered by Theorem 1.1. In fact, there is only a single instance where our proof requires that $\kappa > 2$, namely in the study of the continuous part of the spectrum at (11.26). Perhaps with further analysis one might incorporate the weight 2 case, by a more careful analysis of the residues of the Dirichlet L -functions. The restriction that q is a prime and that $\kappa \leq 14$, $\kappa \neq 12$, means that the cuspforms $f \in H_\kappa(\Gamma_0(q))$ are automatically newforms. It is reasonable to expect that using a more general Petersson formula for newforms (e.g., see [32]) could relax these assumptions, but the arithmetical complexity would be increased.

2. HIGH-LEVEL SKETCH

Here we include an outline of the major steps used in the proof, intended for an expert audience. By approximate functional equations and the Petersson formula, we arrive at

$$(2.1) \quad \mathcal{S} := \sum_{m \ll q} \sum_{n \ll q^{3/2}} \sum_{c \equiv 0 \pmod{q}} \frac{\tau(m)\tau_3(n)}{c\sqrt{mn}} S(m, n; c) J_{\kappa-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

and we wish to show $\mathcal{S} \ll q^{\theta+\varepsilon}$. The hardest case to consider is $m \asymp q$, $n \asymp q^{3/2}$, and $c \asymp \sqrt{mn} \asymp q^{5/4}$, in which case $J_{\kappa-1}(x) \approx 1$. In practice, one needs to treat the two ranges of the Bessel function (i.e., $x \ll 1$ and $x \gg 1$) differently. In this sketch, we focus on the transition region of the J -Bessel function where $x \asymp 1$.

The Weil bound applied to the Kloosterman sum shows $\mathcal{S} \ll q^{7/8+\varepsilon}$, far from $q^{\theta+\varepsilon}$.

The immediate problem with (2.1) is that Voronoi summation applied to m or n leads to a dual sum that is longer than before. The conventional wisdom is that this is a bad move. However, there do not seem to be any other moves available, so it may be necessary to take a loss in the first step. We may attempt to minimize the loss here by opening $\tau(m) = \sum_{m_1 m_2 = m} 1$, supposing $m_1 \leq m_2$ by symmetry, and applying Poisson summation in m_2 modulo c . This leads to

$$\mathcal{S} \approx \sum_{m_1 \ll q^{1/2}} \sum_{\substack{c \equiv 0 \pmod{q} \\ c \asymp q^{5/4}}} \sum_{k \ll q^{3/4}} \sum_{n \asymp q^{3/2}} \frac{\tau_3(n)}{\sqrt{m_1 k n c}} e\left(\frac{m_1 n \bar{k}}{c}\right).$$

Note that the trivial bound now gives $\mathcal{S} \ll q$, so we lost a factor $q^{1/8}$ from going the “wrong way” in Poisson. However, now we may gain from the structure of the arithmetical part by applying the well-known reciprocity formula

$$e\left(\frac{m_1 n \bar{k}}{c}\right) = e\left(-\frac{m_1 n \bar{c}}{k}\right) e\left(\frac{m_1 n}{ck}\right).$$

This effectively switches the roles of c and k , at the expense of introducing the potentially-oscillatory factor $e_{ck}(m_1 n)$ into the weight function. However, when all variables are near their maximal sizes, then this factor is not oscillatory, so we shall ignore it in this sketch.

Side remark. If one applies Voronoi to the sum over m (which is more in line with the previous works on the amplified/mollified fourth moment), then one encounters a shifted

divisor sum of the form $\sum_{m-n=h} \tau_2(m)\tau_3(n)$. Such sums have been considered by various authors, with the most advanced results being the recent work of B. Topaçoğulları [34].

One way to proceed next would be to convert the additive character into Dirichlet characters (modulo k), which has a nice benefit of separating the variables, a key step in [7]. This would lead to a fifth moment of Dirichlet L -functions twisted by Gauss sums, with an averaging over the modulus. One may check that Lindelöf applied to these L -functions only gives $\mathcal{S} \ll q^{1/4+\varepsilon}$ which in a sense gets back to the convexity bound.

Now it is beneficial to apply Voronoi summation in n modulo k (one may view this as opening $\tau_3(n) = \sum_{n_1 n_2 n_3 = n} 1$, and applying Poisson in each n_i). This leads to

$$(2.2) \quad \mathcal{S} \approx \sum_{m_1 \ll q^{1/2}} \sum_{\substack{c \equiv 0 \pmod{q} \\ c \asymp q^{5/4}}} \sum_{k \ll q^{3/4}} \sum_{p \ll q^{3/4}} \frac{\tau_3(p)}{k \sqrt{m_1 p c}} S(p, c \overline{m_1}, k).$$

The trivial bound now is $q^{5/8}$, consistent with saving $q^{1/8}$ in each of the n_i variables, just as we lost $q^{1/8}$ by Poisson in the initial m_2 variable. One could also apply Poisson in m_1 to save another $q^{1/8}$, but then the arithmetical sum becomes a hyper-Kloosterman sum, which increases the difficulty of the problem (N. Pitt has studied this problem [33], but it seems very hard to obtain enough cancellation using this approach). Here we have a Kloosterman sum to which we may apply the Bruggeman-Kuznetsov formula of level m_1 . Using this, we obtain

$$(2.3) \quad \mathcal{S} \approx \sum_{m_1 \ll q^{1/2}} \sum_{\substack{t_j \ll q^\varepsilon \\ \text{level } m_1}} \sum_{\substack{c \equiv 0 \pmod{q} \\ c \asymp q^{5/4}}} \sum_{p_1, p_2, p_3 \ll q^{1/4}} \frac{\nu_j(p_1 c) \nu_j(p_2 p_3)}{\sqrt{p_1 p_2 p_3 c}}.$$

We can essentially write this as

$$(2.4) \quad \mathcal{S} \approx \sum_{m_1 \ll q^{1/2}} \sum_{\substack{t_j \ll q^\varepsilon \\ \text{level } m_1}} \nu_j(1)^2 \frac{\lambda_j(q)}{\sqrt{q}} L(1/2, u_j)^4.$$

Here the scaling on the spectral data is that $\sum_{t_j \ll T} \nu_j(1)^2 \ll T^2 m_1^\varepsilon$. Thus we have converted to a twisted fourth moment of Maass form L -functions, and one can see how $q^{\theta+\varepsilon}$ emerges by bounding $|\lambda_j(q)| \ll q^{\theta+\varepsilon}$, and using a Lindelöf-on-average bound for the spectral fourth moment (which in turn is “easy”, following from the spectral large sieve inequality).

The above discussion implicitly assumed that the p_i are nonzero. The zero frequencies (where some or all $p_i = 0$) turn out to be the “fake” main terms alluded to in the introduction.

To handle these, we compute the weight function explicitly, and evaluate the sums over k, m_1 , and c as zeta quotients. We later bound the integral by moving lines of integration, and apparent poles of the integrand are cancelled by a choice of the weight function in the approximate functional equation. To elaborate on this point, consider an overly-simplified model with a sum of the form $S = \sum_{n \geq 1} \frac{1}{\sqrt{n}} V(\frac{n}{\sqrt{q}})$, where $V(x) = \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} x^{-s} ds$, and $G(s)$ is analytic satisfying $G(0) = 1$, with rapid decay in the imaginary direction. The trivial bound applied to S gives $S = O(q^{1/4})$, using that $V(x) \ll (1+x)^{-100}$. Alternatively, we have $S = \frac{1}{2\pi i} \int_{(1)} \zeta(1/2+s) q^{s/2} \frac{G(s)}{s} ds$, which by shifting contours to the line $\text{Re}(s) = \varepsilon > 0$ gives $S = G(1/2) q^{1/4} + O(q^\varepsilon)$. If $G(1/2) = 0$ (which one is free to assume in the context of the approximate functional equation), then in fact one has an improved bound of $S = O(q^\varepsilon)$. This is the principal idea behind the estimation of the fake main terms. The main difficulty

in practice is that one has a much more complicated sum with multiple variables and weight functions that arise as integral transforms, and it requires significant work to recognize instances of this basic idea. One should also observe that the above method of estimating S is highly reliant on the specific form of the weight function V ; if it were multiplied by a compactly-supported bump function (say one part of a dyadic partition of unity), then one could not deduce $S = O(q^\varepsilon)$ anymore. Since we shall apply dyadic partitions of unity in the forthcoming treatment, for the purposes of estimating these fake main terms, it is crucial to re-assemble the partitions.

The role of the m_1 -variable within the proof has some curious features. In the sketch above up through (2.4), the m_1 variable was hardly used. Precisely, we never applied a summation formula nor obtained any cancellation from this variable. Nor did we use any reciprocity involving m_1 to lower a modulus. However, non-trivial estimations involving m_1 do appear in other parts of the proof. In the evaluation of one type of fake main term in Section 13.7, we evaluate the m_1 -sum similarly to the discussion in the previous paragraph; the lack of pole at $s = 1/2$ amounts to square-root cancellation in this variable. The other location is in estimating the continuous spectrum analog of (2.3) which so far was neglected in this sketch. One may show that the continuous spectrum analog of (2.4) is $O(q^\varepsilon)$ using that the number of cusps on $\Gamma_0(m_1)$ is at most $O(m_1^{1/2+\varepsilon})$. However, on average over m_1 , the number of cusps is $O(m_1^\varepsilon)$, which leads to a bound that saves an additional factor $q^{1/4}$. In a sense, this discussion indicates that the continuous spectrum is smaller in measure *in the level aspect* than the discrete spectrum, on average over the level.

3. PRELIMINARIES

3.1. Petersson Trace Formula. We normalize the Petersson inner product by

$$\langle f, g \rangle = \iint_{\Gamma_0(q) \backslash \mathbb{H}} f(z) \overline{g(z)} y^\kappa \frac{dx dy}{y^2}.$$

For f a Hecke-normalized (holomorphic or Maass) cusp form, one has the estimate

$$q^{1-\varepsilon} \ll_{\kappa, \varepsilon} \langle f, f \rangle \ll_{\kappa, \varepsilon} q^{1+\varepsilon}$$

for any $\varepsilon > 0$, by [18] and [15]. The Petersson trace formula reads

$$\sum_{f \in H_\kappa(q)} w_f \lambda_f(n) \lambda_f(m) = \delta_{n=m} + 2\pi i^{-\kappa} \sum_{c \equiv 0 \pmod{q}} \frac{S(m, n; c)}{c} J_{\kappa-1} \left(\frac{4\pi \sqrt{mn}}{c} \right),$$

where $w_f = \frac{\Gamma(\kappa-1)}{(4\pi)^{\kappa-1} \langle f, f \rangle} = q^{-1+o(1)}$ are the Petersson weights. Define

$$\mathcal{M} = \mathcal{M}(q) = \sum_{f \in H_\kappa(q)} w_f L\left(\frac{1}{2}, f\right)^5.$$

Our main result, Theorem 1.1, is equivalent to

$$(3.1) \quad \mathcal{M} \ll_{\kappa, \varepsilon} q^{\theta+\varepsilon}.$$

3.2. The approximate functional equations. Let κ be a positive even integer, q a prime, and f a Hecke cusp form of weight κ and level q . Put

$$\gamma(s, \kappa) = \pi^{-s} \Gamma\left(\frac{s + \frac{\kappa-1}{2}}{2}\right) \Gamma\left(\frac{s + \frac{\kappa+1}{2}}{2}\right).$$

Let G_i ($i = 1, 2$) be an even entire function decaying rapidly in vertical strips such that $G_i(0) = 1$. Define

$$V_1(x) = \frac{1}{2\pi i} \int_{(1)} \frac{G_1(u)}{u} \frac{\gamma(\frac{1}{2} + u, \kappa)}{\gamma(\frac{1}{2}, \kappa)} x^{-u} du, \quad V_2(x) = \frac{1}{2\pi i} \int_{(1)} \frac{G_2(u)}{u} \frac{\gamma(\frac{1}{2} + u, \kappa)^2}{\gamma(\frac{1}{2}, \kappa)^2} x^{-u} du.$$

If $x \gg q^\varepsilon$ then by shifting the contour of integration arbitrarily far to the right, we obtain that $V_i(x) \ll_{\kappa, A} (xq)^{-A}$. Here and throughout, we view κ as fixed, and q as becoming large. For later use, it will be important to assume $G_i(1/2) = 0$.

Proposition 3.1. *With notation as above, we have*

$$L(\tfrac{1}{2}, f)^2 = 2 \sum_{m=1}^{\infty} \frac{\lambda_f(m) \tau_2(m)}{\sqrt{m}} V\left(\frac{m}{q}\right),$$

where $\tau_2(m)$ is the (two-fold) divisor function, and

$$V(x) = \sum_{(e, q)=1}^{\infty} V_2(e^2 x)/e = \frac{1}{2\pi i} \int_{(1)} \tilde{V}_2(u) \zeta_q(1+2u) x^{-u} du,$$

where $\zeta_q(s) = (1 - q^{-s})\zeta(s)$ is the Riemann zeta function with the q^{th} Euler factor missing.

Proof. By the Hecke relation, we have

$$(3.2) \quad L^2(s, f) = \sum_{m_1, m_2=1}^{\infty} \sum_{\substack{e|(m_1, m_2) \\ (e, q)=1}} \frac{\lambda_f(m_1 m_2 / e^2)}{(m_1 m_2)^s} = \sum_{(e, q)=1}^{\infty} \frac{1}{e^{2s}} \sum_{m=1}^{\infty} \frac{\tau_2(m) \lambda_f(m)}{m^s}.$$

Then from the functional equation $L^2(s, f) \gamma(s, \kappa)^2 q^s =: \Lambda(s, f)^2 = \Lambda(1-s, f)^2$ we get the formula, as in [21, Theorem 5.3]. \square

Proposition 3.2. *Let ε_f be the sign of the functional equation for $L(s, f)$. Then*

$$(3.3) \quad L(\tfrac{1}{2}, f)^3 = (1 + \varepsilon_f)^3 \sum_{\substack{a=1 \\ (a, q)=1}}^{\infty} \frac{\mu(a)}{a^{3/2}} \sum_{n=1}^{\infty} \frac{\lambda_f(na)}{\sqrt{n}} \tau_3(n, F_{a, \sqrt{q}}),$$

where

$$(3.4) \quad \tau_3(n, F_{a, \sqrt{q}}) = \sum_{n_1 n_2 n_3 = n} F_a\left(\frac{n_1}{\sqrt{q}}, \frac{n_2}{\sqrt{q}}, \frac{n_3}{\sqrt{q}}\right),$$

and

$$(3.5) \quad F_a(x_1, x_2, x_3) = \sum_{\substack{e_1, e_2, e_3 \\ (e_1 e_2 e_3, q)=1}} \frac{1}{e_1 e_2 e_3} V_1(ax_1 e_1 e_2) V_1(ax_2 e_1 e_3) V_1(ax_3 e_2 e_3) \\ = \iiint \prod_{i=1}^3 \frac{\gamma(\frac{1}{2} + u_i, \kappa) G(u_i)}{(ax_i)^{u_i} \gamma(\frac{1}{2}, \kappa) u_i} \zeta_q(1 + u_1 + u_2) \zeta_q(1 + u_1 + u_3) \zeta_q(1 + u_2 + u_3) \frac{du_1 du_2 du_3}{(2\pi i)^3}.$$

Remark. One may easily check that

$$(3.6) \quad x_1^{j_1} x_2^{j_2} x_3^{j_3} \frac{\partial^{j_1+j_2+j_3}}{\partial x_1^{j_1} \partial x_2^{j_2} \partial x_3^{j_3}} F_a(x_1, x_2, x_3) \ll_{j_1, j_2, j_3, A, \varepsilon} \prod_{i=1}^3 (ax_i)^{-\varepsilon} (1 + ax_i)^{-A}.$$

In the terminology introduced later in Section 5, the property (3.6) means that F_a satisfies the same derivative bounds as an X -inert function with $X \ll q^\varepsilon$, in the region $x_i \gg q^{-1/2}$, for all i . Similar derivative bounds hold for $V(x)$.

Proof. Using the approximate functional equation for each $L(1/2, f)$, and the Hecke relations, we obtain

$$\begin{aligned} L\left(\frac{1}{2}, f\right)^3 &= \sum_{\substack{e_1, e_2 \\ (e_1 e_2, q)=1}} \frac{(1 + \varepsilon_f)^3}{e_1} \sum_{\substack{n_1, n_2, n_3 \\ e_2 | (n_1 n_2, n_3)}} \frac{\lambda_f\left(\frac{n_1 n_2 n_3}{e_2^2}\right)}{\sqrt{n_1 n_2 n_3}} V_1\left(\frac{n_1 e_1}{\sqrt{q}}\right) V_1\left(\frac{n_2 e_1}{\sqrt{q}}\right) V_1\left(\frac{n_3}{\sqrt{q}}\right) \\ &= \sum_{\substack{e_1, e_2 \\ (e_1 e_2, q)=1}} \frac{(1 + \varepsilon_f)^3}{e_1 e_2} \sum_{f_1 f_2 = e_2} \sum_{\substack{n_1, n_2, n_3 \\ (n_1, f_2)=1}} \frac{\lambda_f(n_1 n_2 n_3)}{\sqrt{n_1 n_2 n_3}} V_1\left(\frac{n_1 e_1 f_1}{\sqrt{q}}\right) V_1\left(\frac{n_2 e_1 f_2}{\sqrt{q}}\right) V_1\left(\frac{n_3 f_1 f_2}{\sqrt{q}}\right). \end{aligned}$$

Using Möbius inversion to detect the coprimality condition with $\sum_{a|(n_1, f_2)} \mu(a)$, re-ordering the summations, and renaming the summation variables gives the more symmetric form

$$\begin{aligned} L\left(\frac{1}{2}, f\right)^3 &= (1 + \varepsilon_f)^3 \sum_{\substack{a=1 \\ (a, q)=1}}^{\infty} \frac{\mu(a)}{a^{3/2}} \sum_{\substack{e_1, e_2, e_3 \\ (e_1 e_2 e_3, q)=1}} \frac{1}{e_1 e_2 e_3} \\ &\quad \times \sum_{n_1, n_2, n_3} \frac{\lambda_f(an_1 n_2 n_3)}{\sqrt{n_1 n_2 n_3}} V_1\left(\frac{an_1 e_1 e_2}{\sqrt{q}}\right) V_1\left(\frac{an_2 e_1 e_3}{\sqrt{q}}\right) V_1\left(\frac{an_3 e_2 e_3}{\sqrt{q}}\right). \end{aligned}$$

This is seen to be equivalent to (3.3). \square

Now apply Propositions 3.1 and 3.2 to \mathcal{M} . There is a nice simplification in Proposition 3.2, whereby we may replace $1 + \varepsilon_f$ by 2, because if $\varepsilon_f = -1$, then $L(1/2, f)^2 = 0$ anyway. Applying the Petersson trace formula then yields

$$\frac{1}{16} \mathcal{M} = \mathcal{D} + 2\pi i^{-\kappa} \mathcal{S},$$

where \mathcal{D} is the diagonal term, and

$$(3.7) \quad \mathcal{S} = \sum_{(a, q)=1}^{\infty} \frac{\mu(a)}{a^{3/2}} \sum_{c \equiv 0 \pmod{q}} \sum_{n, m} \frac{\tau_2(m) \tau_3(n, F_{a, \sqrt{q}}) S(m, na; c)}{c \sqrt{mn}} J_{\kappa-1}\left(\frac{4\pi \sqrt{mna}}{c}\right) V\left(\frac{m}{q}\right).$$

It is easy to bound the diagonal term.

Lemma 3.3. *We have*

$$\mathcal{D} \ll_{\varepsilon} q^{\varepsilon}.$$

This follows easily from the fact that the functions $V_1(y)$ and $V_2(y)$ decay rapidly as $y \rightarrow \infty$, and using the bound $J_{\kappa-1}(x) \ll x$ for $\kappa \geq 2$.

Proving Theorem 1.1 is reduced to showing $\mathcal{S} \ll q^{\theta+\varepsilon}$. We will return to \mathcal{S} in Section 6. Meanwhile, Sections 4 and 5, which are self-contained, develop some material necessary for our manipulations of \mathcal{S} .

4. KLOOSTERMAN SUMS AND BRUGGEMAN-KUZNETSOV SUMMATION FORMULA

4.1. **Motivation.** We seek a formula for the sum

$$\sum_{\substack{(c,N)=1 \\ c \equiv 0 \pmod{q}}} S(\overline{N}m, n; c) f(c),$$

where f is a smooth function on the positive reals with sufficient decay. In this section we show that this sum of Kloosterman sums has a spectral decomposition since it can be realized as the Kloosterman sum associated to the pair of cusps $\infty, 1/q$ for the group $\Gamma_0(Nq)$. This precise form of Kloosterman sum seems to not appear in the standard sources in the literature, such as [20] or [8]. Moreover, we show that the scaling matrices implicit in the definition of the Kloosterman sum as well as in the Fourier expansion at certain cusps (which we call Atkin-Lehner cusps) may be chosen to be Atkin-Lehner operators.

We record the Bruggeman-Kuznetsov formula in Section 4.5. We will specify the choice of a basis of automorphic forms in this expansion in Section 4.7.

4.2. **Cusps, scaling matrices, and Kloosterman sums.** We mostly follow the notation of [20]. Let N be a positive integer and $\Gamma = \Gamma_0(N)$. An element $\mathfrak{a} \in \mathbb{P}^1(\mathbb{Q})$ is called a cusp. Two cusps \mathfrak{a} and \mathfrak{a}' are equivalent under Γ if there is a $\gamma \in \Gamma$ satisfying $\mathfrak{a}' = \gamma\mathfrak{a}$. Let \mathfrak{a} be a cusp and $\Gamma_{\mathfrak{a}} = \{\gamma \in \Gamma : \gamma\mathfrak{a} = \mathfrak{a}\}$ be the stabilizer of the cusp \mathfrak{a} in Γ . A matrix $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{R})$, satisfying

$$(4.1) \quad \sigma_{\mathfrak{a}}\infty = \mathfrak{a}, \quad \text{and} \quad \sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\}$$

is called a scaling matrix for the cusp \mathfrak{a} .

Remark. For two equivalent cusps \mathfrak{a} and $\mathfrak{a}' = \gamma\mathfrak{a}$, with $\gamma \in \Gamma$, the stabilizers of the cusps are conjugate subgroups in Γ , namely $\Gamma_{\mathfrak{a}'} = \gamma\Gamma_{\mathfrak{a}}\gamma^{-1}$. Furthermore, $\sigma_{\mathfrak{a}'} = \gamma\sigma_{\mathfrak{a}}$ is a scaling matrix for the cusp \mathfrak{a}' .

Remark. The matrix $\sigma_{\mathfrak{a}}$ is not uniquely defined by the above properties: for any $x \in \mathbb{R}$,

$$\sigma_{\mathfrak{a}} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

also satisfies (4.1). The choice of the scaling matrix $\sigma_{\mathfrak{a}}$ is important in what follows.

Definition 4.1. Let f be a Maass form for the group Γ . The Fourier coefficients of f at a cusp \mathfrak{a} , relative to a choice of $\sigma_{\mathfrak{a}}$, and denoted $\rho_f(\sigma_{\mathfrak{a}}, n)$, are defined by

$$(4.2) \quad f(\sigma_{\mathfrak{a}}z) = \sum_{n \neq 0} \rho_f(\sigma_{\mathfrak{a}}, n) W_{\frac{1}{2}+it_j}(nz),$$

where

$$W_s(x + iy) = 2\sqrt{|y|} K_{s-\frac{1}{2}}(2\pi|y|) e(x).$$

Remark. If $\sigma_{\mathfrak{a}}$ is replaced with $\sigma_{\mathfrak{a}} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, then the Fourier coefficients relative to the new scaling matrix are given by

$$\rho_f(\sigma_{\mathfrak{a}} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, n) = e(n\alpha) \rho_f(\sigma_{\mathfrak{a}}, n).$$

If the Fourier coefficients of f at a cusp \mathfrak{a} are multiplicative with respect to $\sigma_{\mathfrak{a}}$, then for another scaling matrix as above, the Fourier coefficients will typically not be multiplicative. We found the reference [13] useful for its discussions in this context.

Fourier coefficients at equivalent cusps, however, behave more predictably. If $\mathfrak{a}' = \gamma\mathfrak{a}$ and we choose the scaling matrix as $\sigma_{\mathfrak{a}'} = \gamma\sigma_{\mathfrak{a}}$, then due to Γ -invariance of f , we have

$$\rho_f(\sigma_{\mathfrak{a}'}, n) = \rho_f(\sigma_{\mathfrak{a}}, n).$$

When the scaling matrix $\sigma_{\mathfrak{a}}$ is understood, we may write $\rho_f(\sigma_{\mathfrak{a}}, n) = \rho_{\mathfrak{a},f}(n)$.

We now define Kloosterman sums with respect to a pair of cusps. Even though we will not need characters in our work, we state the definition with nebentypus, since we expect this will be useful in other contexts. Let χ be a Dirichlet character modulo N . We extend χ to Γ via

$$\begin{aligned} \chi : \Gamma &\longrightarrow S^1 \\ \gamma = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} &\mapsto \chi(d). \end{aligned}$$

If χ is an even character, it can be seen as a multiplier system on Γ with weight 0.

Definition 4.2. *The cusp \mathfrak{a} is called singular for the character χ if χ is trivial on $\Gamma_{\mathfrak{a}}$. For \mathfrak{a} and \mathfrak{b} singular cusps for χ , we define the Kloosterman sum associated to $\mathfrak{a}, \mathfrak{b}$ and χ with modulus c as*

$$(4.3) \quad S_{\mathfrak{a}\mathfrak{b}}(m, n; c; \chi) = \sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\infty} \setminus \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}} / \Gamma_{\infty}} \overline{\chi(\sigma_{\mathfrak{a}} \gamma \sigma_{\mathfrak{b}}^{-1})} e\left(\frac{am + dn}{c}\right).$$

Remarks. This definition of the Kloosterman sum slightly differs from that of [20, (2.23)], in that the roles of m and n are reversed. Also note that one requires \mathfrak{a} and \mathfrak{b} to be singular cusps with respect to χ for the sum to be well-defined, so in particular χ is even.

From Definition 4.2 one may directly deduce

$$(4.4) \quad S_{\mathfrak{a}\mathfrak{b}}(m, n; c; \chi) = \overline{S_{\mathfrak{b}\mathfrak{a}}(n, m; c; \chi)}.$$

This corrects a formula in [20, p.48] which omitted the complex conjugation. In many important cases the Kloosterman sum is real, e.g. see Theorem 4.6 below.

Definition 4.3. *Let the set of allowed moduli be defined as*

$$(4.5) \quad \mathcal{C}_{\mathfrak{a}\mathfrak{b}} = \left\{ \gamma > 0 : \begin{pmatrix} * & * \\ \gamma & * \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}} \right\}.$$

Notice that if $\gamma \notin \mathcal{C}_{\mathfrak{a}\mathfrak{b}}$ the Kloosterman sum of modulus γ is an empty sum.

The definition (4.3) is a natural one, as it occurs in the Fourier expansion of the Poincaré series which is defined as

$$(4.6) \quad P_n^{\mathfrak{a}}(z, s; \chi) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \setminus \Gamma} \overline{\chi(\gamma)} \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma z)^s e(n \sigma_{\mathfrak{a}}^{-1} \gamma z).$$

For $n \neq 0$, $P_n^{\mathfrak{a}}(z, s; \chi) \in L^2(\Gamma \backslash \mathbb{H}, \chi)$ (to be clear, it transforms by $f(\gamma z) = \chi(\gamma) f(z)$).

Remark. The Kloosterman sum associated to the pair of cusps $\mathfrak{a}, \mathfrak{b}$ depends on the choice of pair of scaling matrices $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$ (so it might be better to denote it as $S_{\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}}}(m, n; c)$). If one changes the choice of the scaling matrix, the Kloosterman sum also changes by

$$(4.7) \quad S_{\sigma_{\mathfrak{a}} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \sigma_{\mathfrak{b}} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}}(m, n; c) = e(-\alpha m + \beta n) S_{\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}}}(m, n; c).$$

This corrects a formula of [20, p.48] which has α in place of our $-\alpha$. However, changing the cusp \mathfrak{a} to an equivalent one does not alter the Kloosterman sum, if one also changes the scaling matrices accordingly. Indeed, if $\mathfrak{a}' = \gamma_1 \mathfrak{a}$ and $\mathfrak{b}' = \gamma_2 \mathfrak{b}$ for $\gamma_1, \gamma_2 \in \Gamma$, then

$$S_{\sigma_{\mathfrak{a}'}, \sigma_{\mathfrak{b}'}}(m, n; c; \chi) = S_{\gamma_1 \sigma_{\mathfrak{a}}, \gamma_2 \sigma_{\mathfrak{b}}}(m, n; c; \chi).$$

If one applies the Bruggeman-Kuznetsov formula (see Section 4.5) to sums of Kloosterman sums associated to the choice of scaling matrices $\sigma_{\mathbf{a}}, \sigma_{\mathbf{b}}$, then the Fourier coefficients at cusps of automorphic forms on the spectral side must also be computed using the same scaling matrices.

4.3. Atkin-Lehner cusps and scaling matrices. Assume that $N = rs$ with $(r, s) = 1$. We call a cusp of the form $\mathbf{a} = 1/r$ (with $(r, s) = 1$) an *Atkin-Lehner cusp*. The stabilizer of such a cusp is given as

$$(4.8) \quad \Gamma_{1/r} = \left\{ \pm \begin{pmatrix} 1 - Nt & st \\ -rNt & 1 + Nt \end{pmatrix} : t \in \mathbb{Z} \right\} = \left\langle \pm \begin{pmatrix} 1 - N & s \\ -rN & 1 + N \end{pmatrix} \right\rangle.$$

In particular we see that any even Dirichlet character $\chi \pmod{N}$ is singular at such a cusp.

Definition 4.4. Let $N = rs$ with $(r, s) = 1$ as above, and put

$$(4.9) \quad W = \begin{pmatrix} xs & y \\ zN & ws \end{pmatrix},$$

with $\det(W) = s$. On automorphic forms of weight κ , the Atkin-Lehner operator is defined by $(f|_W)(z) = \det(W)^{\frac{\kappa}{2}} j(W, z)^{-\kappa} f(Wz)$.

The Atkin-Lehner operator is independent of the choices of x, y, z, w . Since the matrix W normalizes Γ , it preserves automorphic forms on Γ . Furthermore, an Atkin-Lehner operator commutes with all the Hecke operators and hence a newform will be an eigenfunction of all the Atkin-Lehner operators. The cusps of the form $\mathbf{a} = 1/r$ with $(r, s) = 1$ are precisely those that are equivalent to ∞ under an Atkin-Lehner operator, which justifies naming them Atkin-Lehner cusps.

The matrix can be normalized to have determinant 1, without changing the operator, via

$$\frac{1}{\sqrt{s}} W = \begin{pmatrix} x\sqrt{s} & y/\sqrt{s} \\ zr\sqrt{s} & w\sqrt{s} \end{pmatrix} = \begin{pmatrix} x & y \\ zr & ws \end{pmatrix} \begin{pmatrix} \sqrt{s} & 0 \\ 0 & 1/\sqrt{s} \end{pmatrix}.$$

Here the determinant condition is $xws - rzy = 1$. We have the freedom to choose $x = z = 1$, and then if \bar{s} is any integer that satisfies $s\bar{s} \equiv 1 \pmod{r}$, put $w = \bar{s}$ and $y = (\bar{s}s - 1)/r$. Therefore the matrix

$$(4.10) \quad \sigma_{1/r} = \tau_r \nu_s \quad \text{with} \quad \tau_r = \begin{pmatrix} 1 & (\bar{s}s - 1)/r \\ r & \bar{s}s \end{pmatrix}, \quad \nu_s = \begin{pmatrix} \sqrt{s} & 0 \\ 0 & 1/\sqrt{s} \end{pmatrix},$$

is an acceptable choice for an Atkin-Lehner operator. Note $\tau_r \in \Gamma_0(r)$ (in particular, it has integer entries and determinant 1). From the theory of Atkin-Lehner operators, a newform f will satisfy

$$(4.11) \quad f(\sigma_{1/r} z) = \pm f(z).$$

One may check directly that $\sigma_{1/r}$ also satisfies the conditions in (4.1), i.e. it is a scaling matrix for the cusp $1/r$. If f is a Maass form satisfying (4.11) then its Fourier coefficients at the cusp \mathbf{a} with respect to the choice (4.10) for its scaling matrix has the Fourier coefficients

$$\rho_f(\sigma_{1/r}, n) = \pm \rho_{\infty, f}(n).$$

Therefore, with this choice of the scaling matrix the Fourier coefficients satisfy the multiplicative Hecke relations. This will be made more explicit in Section 4.7.

4.4. Kloosterman sums using Atkin-Lehner scaling.

Proposition 4.5. *Let $N = rs$ with $(r, s) = 1$, and choose $\sigma_{1/r}$ as in (4.10). Then the set of allowed moduli for the pair of cusps $\infty, \frac{1}{r}$ is*

$$(4.12) \quad \mathcal{C}_{\infty, 1/r} = \{ \gamma = c\sqrt{s} > 0 : c \equiv 0 \pmod{r}, (c, s) = 1 \},$$

and for such $\gamma = c\sqrt{s} \in \mathcal{C}_{\infty, 1/r}$, the Kloosterman sum to modulus γ is given by

$$(4.13) \quad S_{\infty, 1/r}(m, n; c\sqrt{s}) = S(\overline{s}m, n; c),$$

where the S on the right denotes an ordinary Kloosterman sum.

Remark. This is the exact same Kloosterman sum that is in (14.8) of [30], but differs from the computation in Section 4.2 of [19] by an additive character. That difference is exactly due to the choice of the scaling matrix. Motohashi's choice of scaling matrix results in Fourier coefficients at the cusp $1/r$ which are multiplicative if the form is a newform.

Below we give the more general Kloosterman sum that is associated to the pair of Atkin-Lehner cusps $1/r_1$ and $1/r_2$ with $N = r_1s_1 = r_2s_2$ and $(r_1, s_1) = (r_2, s_2)$ and also with a character $\chi \pmod{N}$. We provide the proof for completeness, since Kloosterman sums with characters were not computed in [30], and since we believe the evaluations should prove useful for other works.

Theorem 4.6. *Let $N = pquv$ with p, q, u, v all pairwise coprime. Put $r_1 = pu, s_1 = qv$ and $r_2 = pv, s_2 = qu$. The set of allowed moduli for the pair of cusps $1/r_1, 1/r_2$ is given as*

$$(4.14) \quad \mathcal{C}_{1/r_1, 1/r_2} = \{ \gamma = c\sqrt{uv} > 0 : c \equiv 0 \pmod{pq}, (c, uv) = 1 \}.$$

Let χ be a character modulo N , and factor it as $\chi = \chi_p \chi_q \chi_u \chi_v$ where χ_p is a character modulo p , χ_q is a character modulo q , etc. The Kloosterman sum for this pair of cusps and character χ with modulus $\gamma = c\sqrt{uv} \in \mathcal{C}_{1/r_1, 1/r_2}$ is given as

$$(4.15) \quad S_{1/r_1, 1/r_2}(m, n; c\sqrt{uv}; \chi) = \mathfrak{f} \overline{\chi_u}(c) \chi_v(c) \sum_{\substack{a, d \pmod{c} \\ ad \equiv 1 \pmod{c}}} \overline{\chi_p}(d) \overline{\chi_q}(a) e\left(\frac{a\overline{u}vm + dn}{c}\right),$$

where

$$(4.16) \quad \mathfrak{f} = \mathfrak{f}(p, q, u, v, \chi) = \chi_v(-1) \overline{\chi_p \chi_v}(u) \chi_q \chi_u(v) \chi_u(pq) \overline{\chi_v}(pq).$$

Remark. One may directly verify that the explicit formula given by (4.15) and (4.16) also satisfies (4.4) (recall that χ is even), which is a nice consistency check.

Proof. Our proof closely follows that in [30]. Consider the double coset $\sigma_{1/r_1}^{-1} \Gamma \sigma_{1/r_2}$, and recall the definitions of τ_r and ν_s from (4.10). We firstly claim

$$(4.17) \quad \tau_{r_1}^{-1} \Gamma \tau_{r_2} = \left\{ \begin{pmatrix} xv & yq \\ zp & wu \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : x, y, z, w \in \mathbb{Z} \right\}.$$

By reducing the entries of the product of matrices implicit in the left hand side of (4.17) modulo p, q, u , and v respectively, one sees that in each case the lower-left, upper-right, lower-right, and upper-left entries vanish respectively. Hence the left hand side of (4.17) is contained in the set on the right hand side. For the opposite inclusion, we have

$$\tau_{r_1} \begin{pmatrix} xv & yq \\ zp & wu \end{pmatrix} \tau_{r_2}^{-1} = \begin{pmatrix} 1 & \frac{s_1 \overline{s_1} - 1}{r_1} \\ r_1 & s_1 \overline{s_1} \end{pmatrix} \begin{pmatrix} xv & yq \\ zp & wu \end{pmatrix} \begin{pmatrix} s_2 \overline{s_2} & \frac{1 - s_2 \overline{s_2}}{r_2} \\ -r_2 & 1 \end{pmatrix}.$$

Again by reducing modulo p, q, u and v (the reader may find it easiest to reduce prior to performing matrix multiplication), one obtains an upper triangular matrix in each case, whence the product is an element of $\Gamma_0(N) = \Gamma_0(pquv)$.

Multiplying with the width-normalizing matrices ν_s , we get

$$\sigma_{1/r_1}^{-1} \Gamma \sigma_{1/r_2} = \left\{ \begin{pmatrix} x\sqrt{uv} & y/\sqrt{uv} \\ zpq\sqrt{uv} & w\sqrt{uv} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : x, y, z, w \in \mathbb{Z} \right\}.$$

The determinant condition reads as $xwuv - zpqy = 1$, and for this to be satisfied one needs $(z, uv) = 1$. This shows that (4.14) indeed gives the allowable set of moduli.

Next we wish to decompose this double coset according to the action of Γ_∞ on both the left and right, as in [20, Theorem 2.7]. A full set of coset representatives for $\Gamma_\infty \backslash \sigma_{\frac{1}{r_1}}^{-1} \Gamma \sigma_{\frac{1}{r_2}} / \Gamma_\infty$ with a given lower-left entry $zpq\sqrt{uv}$ is given by

$$(4.18) \quad (\Gamma_\infty \backslash \sigma_{\frac{1}{r_1}}^{-1} \Gamma \sigma_{\frac{1}{r_2}} / \Gamma_\infty) \cap \left\{ \begin{pmatrix} * & * \\ zpq\sqrt{uv} & * \end{pmatrix} \right\} \\ = \left\{ \begin{pmatrix} x\sqrt{uv} & * \\ zpq\sqrt{uv} & w\sqrt{uv} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : \begin{array}{l} x, w \in (\mathbb{Z}/zpq\mathbb{Z})^* \\ xwuv \equiv 1 \pmod{zpq} \end{array} \right\}.$$

Here the condition $xwuv \equiv 1 \pmod{zpq}$ determines w in terms of x , and automatically implies $(xw, zpg) = 1$.

Because of the presence of a character, we need to know the lower-right entry of an element of Γ in terms of the integers x, w, z from this double coset. Given $\rho = \begin{pmatrix} x\sqrt{uv} & y/\sqrt{uv} \\ zpq\sqrt{uv} & w\sqrt{uv} \end{pmatrix}$, we compute the lower-right entry of $\begin{pmatrix} * & * \\ * & d \end{pmatrix} = \sigma_{\frac{1}{r_1}} \rho \sigma_{\frac{1}{r_2}}^{-1}$ by brute-force calculation as

$$d = \frac{(1 - s_2 \bar{s}_2)}{r_2} (puxv + s_1 \bar{s}_1 zp) + puqy + us_1 \bar{s}_1 w \\ = (1 - qu\bar{q}u)(ux + q\bar{q}vz) + puqy + uqv\bar{q}vw.$$

Reducing this in each of the moduli p, q, u, v , we obtain

$$\begin{aligned} d &\equiv wu \pmod{p}, & d &\equiv ux \pmod{q}, \\ d &\equiv z\bar{v} \pmod{u}, & d &\equiv ypuq \equiv -\bar{z}p\bar{q}puq \equiv -\bar{z}u \pmod{v}. \end{aligned}$$

Alternatively, one may reduce the matrices prior to the matrix multiplication.

The Kloosterman sum is then given by

$$S_{\frac{1}{r_1}, \frac{1}{r_2}}(m, n; zpq\sqrt{uv}; \chi) = \sum_{\begin{pmatrix} x\sqrt{uv} & * \\ zpq\sqrt{uv} & w\sqrt{uv} \end{pmatrix} \in \Gamma_\infty \backslash \sigma_{\frac{1}{r_1}}^{-1} \Gamma \sigma_{\frac{1}{r_2}} / \Gamma_\infty} \bar{\chi}_p(uw) \bar{\chi}_q(ux) \bar{\chi}_u(z\bar{v}) \bar{\chi}_v(-\bar{z}u) e\left(\frac{xm + wn}{zpq}\right).$$

Using (4.18) and a change of variables $x \mapsto x\bar{w}\bar{v}$, and with some simplifications, we obtain (4.15). \square

Examples. Specializing Theorem 4.6 to particular cusps, we obtain

$$(4.19) \quad S_{\infty, 0}(m, n; c\sqrt{N}; \chi) = \bar{\chi}(c) S(\bar{N}m, n; c),$$

with $(c, N) = 1$. More generally, we have

$$(4.20) \quad S_{\infty, \frac{1}{r}}(m, n; c; \chi) = \bar{\chi}_r(s) \chi_s(r) \bar{\chi}_s(c) \sum_{\substack{a, d \pmod{c} \\ ad \equiv 1 \pmod{c}}} e\left(\frac{a\bar{s}m + dn}{c}\right) \bar{\chi}_r(d),$$

with $r|c$ and $(c, s) = 1$, and additionally

$$(4.21) \quad S_{0, \frac{1}{r}}(m, n; c; \chi) = \chi_r(-1) \chi_s(r) \overline{\chi_r}(s) \chi_r(c) \sum_{\substack{a, d \pmod{c} \\ ad \equiv 1 \pmod{c}}} e\left(\frac{a\overline{r}m + dn}{c}\right) \overline{\chi_s}(a),$$

with $s|c$ and $(c, r) = 1$. These formulas should be contrasted with [19, p.58] or [9, Lemma 4.3], which use a different choice of scaling matrices.

In (4.19) the occurrence of the factor $\overline{\chi}(c)$ with the *modulus* of the Kloosterman sum is a nice feature of the pair of cusps $\infty, 0$ as opposed to the case

$$(4.22) \quad S_{\infty, \infty}(m, n; c; \chi) = \sum_{ad \equiv 1 \pmod{c}} e\left(\frac{am + dn}{c}\right) \overline{\chi}(d).$$

Remark. Here we would like to contrast the above discussion with that of [5] where Blomer and Milićević study sums of Kloosterman sums weighted by an arbitrary function $f(c)$ on $(\mathbb{Z}/q\mathbb{Z})^\times$. Their method makes use of $S_{\infty, \infty}(m, n; c; \chi)$ as well, but they require the use of twisted multiplicativity of Kloosterman sums, and the fact that $S_{\infty, \infty}(m, 0; c; \chi)$ is a Gauss sum of the character χ with twist m , in order to obtain the desired form $\chi(c)S(m, n; c)$.

On the other hand, using (4.19), we have

$$(4.23) \quad \begin{aligned} \sum_{c \equiv a \pmod{q}} S(m, n; c) f(c) &= \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \chi(a) \sum_{(c, q)=1} \overline{\chi}(c) S(\overline{q}qm, n; c) f(c) \\ &= \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \chi(a) \sum_{(c, q)=1} S_{\infty, 0}(qm, n; c\sqrt{q}; \chi) f(c). \end{aligned}$$

The sum over $(c, q) = 1$ is the set of all allowed moduli for the group $\Gamma_0(q)$ and the $\infty, 0$ cusp pair, so one may apply the Bruggeman-Kuznetsov formula to this type of sum without any further manipulations.

For one final example, which will be useful later, if χ is the principal character (whence we do not display it in the notation), we have

$$(4.24) \quad \sum_{\substack{(c, s)=1 \\ c \equiv 0 \pmod{r}}} S(\overline{s}m, n; c) f(c) = \sum_{\gamma \in \mathcal{C}_{\infty, 1/r}} S_{\infty, 1/r}(m, n; \gamma) f\left(\frac{\gamma}{\sqrt{s}}\right),$$

where $\Gamma = \Gamma_0(rs)$.

4.5. Bruggeman-Kuznetsov formula. We record the spectral expansion of a sum of Kloosterman sums in a spectral basis of the space $L^2(\Gamma_0(N), \chi)$. Let $\{u_j\}$ be a basis of cusp forms. Assume that u_j is an eigenfunction of the Laplace-Beltrami operator with eigenvalue $\frac{1}{4} + t_j^2$. Call t_j the spectral parameter of u_j . Define $\rho_{aj}(n) = \rho_{u_j}(\sigma_a, n)$ as in (4.2); note that our choice of σ_a , in practice, will be an Atkin-Lehner operator.

Likewise, write the Fourier expansion of the Eisenstein series as

$$(4.25) \quad E_c(\sigma_a z, u; \chi) = \delta_{ac} y^u + \rho_{ac}(0, u, \chi) y^{1-u} + \sum_{n \neq 0} \rho_{ac}(n, u, \chi) W_u(nz).$$

Consulting [20, Theorem 3.4], we have

$$(4.26) \quad \rho_{\text{ac}}(n, u, \chi) = \begin{cases} \phi_{\text{ac}}(n, u, \chi) \frac{\pi^u}{\Gamma(u)} |n|^{u-1}, & \text{if } n \neq 0 \\ \delta_{\text{ac}} y^u + \phi_{\text{ac}}(u, \chi) y^{1-u}, & \text{if } n = 0, \end{cases}$$

where

$$(4.27) \quad \phi_{\text{ac}}(n, u, \chi) = \sum_{\substack{(\gamma, \delta) \text{ such that} \\ \rho = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in \Gamma_{\infty} \setminus \sigma_{\mathbf{c}}^{-1} \Gamma \sigma_{\mathbf{a}} / \Gamma_{\infty}}} \bar{\chi}(\sigma_{\mathbf{c}} \rho \sigma_{\mathbf{a}}^{-1}) \frac{1}{\gamma^{2u}} e\left(\frac{n\delta}{\gamma}\right) = \sum_{\gamma \in \mathcal{C}_{\mathbf{ca}}} \frac{S_{\mathbf{ca}}(0, n; \gamma, \chi)}{\gamma^{2u}},$$

and $\phi_{\text{ac}}(u, \chi) = \phi_{\text{ac}}(0, u, \chi)$. Note that our ordering of the cusps in the notation $\rho_{\text{ac}}, \phi_{\text{ac}}$ is reversed from that of [20], and also that [20, (3.22)] should have $\mathcal{S}_{\text{ac}}(n, 0; c)$ in place of $\mathcal{S}_{\text{ac}}(0, n; c)$ to be consistent with [20, (2.23)]. In case χ is principal, we shall drop it from the notation. We give an explicit computation of $\phi_{\text{ac}}(n, u)$ with (4.64) (see also (4.65)) below.

For aesthetic purposes, define as in [20, (8.5), (8.6)]

$$(4.28) \quad \nu_{\text{aj}}(n) = \left(\frac{4\pi|n|}{\cosh(\pi t_j)} \right)^{\frac{1}{2}} \rho_{\text{aj}}(n), \quad \nu_{\text{ac}}(n, u, \chi) = (4|n|\Gamma(u)\Gamma(1-u))^{\frac{1}{2}} \rho_{\text{ac}}(n, u; \chi).$$

Let $g \in H_k(N, \chi)$, that is, let g be a holomorphic level N weight k modular cusp form with nebentypus χ . Define the Fourier expansion of g at a cusp \mathbf{a} by

$$g(\sigma_{\mathbf{a}} z) = \sum_{n=1}^{\infty} \rho_{\text{ag}}(n) n^{\frac{k-1}{2}} e(nz).$$

Also define (see [20, (9.42)]), and note that we already extracted $m^{\frac{k-1}{2}}$ in the definition of ρ_{ag})

$$(4.29) \quad \nu_{\text{ag}}(n) = \left(\frac{\pi^{-k} \Gamma(k)}{4^{k-1}} \right)^{1/2} \rho_{\text{ag}}(n).$$

With the notation as above, let us define for nonzero m and n ,

$$(4.30) \quad \mathcal{K} = \sum_{\gamma \in \mathcal{C}_{\mathbf{a}, \mathbf{b}}} S_{\mathbf{ab}}(m, n; \gamma; \chi) \phi(\gamma).$$

We then quote the literature for a formula for this sum. Many authors state the Bruggeman-Kuznetsov formula with a weight function of the form $\gamma^{-1} F(\frac{4\pi\sqrt{mn}}{\gamma})$ in place of $\phi(\gamma)$, which amounts to the substitution $F(t) = \frac{4\pi\sqrt{mn}}{t} \phi(\frac{4\pi\sqrt{mn}}{t})$.

Theorem 4.7 ([20] Chapter 9). *Let \mathcal{K} be as in (4.30). Assuming ϕ is smooth with compact support on $(0, \infty)$, we have*

$$\mathcal{K} = \mathcal{K}_d + \mathcal{K}_c + \mathcal{K}_h.$$

Here $\mathcal{K}_h = 0$ if $mn < 0$ and otherwise,

$$(4.31) \quad \mathcal{K}_h = \sum_{k > 0, \text{ even}} \phi_h(k) i^k \sum_{g \in H_k(N, \chi)} \nu_{\text{ag}}(m) \overline{\nu_{\text{bg}}(n)}.$$

The discrete spectrum contribution is given as

$$(4.32) \quad \mathcal{K}_d = \sum_{t_j} \phi_{\pm}(t_j) \nu_{\text{aj}}(m) \overline{\nu_{\text{bj}}(n)},$$

where the summation is over the spectral parameters t_j of a chosen orthonormal basis of cusp forms $\{u_j\}_j$. The continuous spectrum contribution is

$$(4.33) \quad \mathcal{K}_c = \sum_{\mathfrak{c} \text{ singular for } \chi} \frac{1}{4\pi} \int_{-\infty}^{\infty} \phi_{\pm}(t) \nu_{\mathfrak{ac}}(m, \tfrac{1}{2} + it, \chi) \overline{\nu_{\mathfrak{bc}}(n, \tfrac{1}{2} + it, \chi)} dt,$$

where the choice ϕ_+ versus ϕ_- depends on whether $mn > 0$ or $mn < 0$.

Here the integral transform for ϕ_h is given as

$$(4.34) \quad \phi_h(k) = \int_0^{\infty} J_{k-1}(x) \frac{4\pi\sqrt{mn}}{x} \phi\left(\frac{4\pi\sqrt{mn}}{x}\right) \frac{dx}{x} = (J_{k-1} * (x \cdot \phi))(4\pi\sqrt{mn}).$$

With

$$B_{2it}^+(x) = \frac{i}{2 \sinh(\pi t)} (J_{2it}(x) - J_{-2it}(x)),$$

then

$$(4.35) \quad \phi_+(t) = \int_0^{\infty} B_{2it}^+(x) \frac{4\pi\sqrt{mn}}{x} \phi\left(\frac{4\pi\sqrt{mn}}{x}\right) \frac{dx}{x} = (B_{2it}^+ * (x \cdot \phi))(4\pi\sqrt{mn}).$$

Similarly, with

$$B_{2it}^-(x) = \frac{2}{\pi} \cosh(\pi t) K_{2it}(x),$$

we have

$$(4.36) \quad \phi_-(t) = \int_0^{\infty} B_{2it}^-(x) \frac{4\pi\sqrt{|mn|}}{x} \phi\left(\frac{4\pi\sqrt{|mn|}}{x}\right) \frac{dx}{x} = (B_{2it}^- * (x \cdot \phi))(4\pi\sqrt{|mn|}).$$

Remarks. Here we have implemented some corrections of [20] noted by Blomer, Harcos, and Michel [3]. Moreover, the right hand side slightly differs from the formulas in [20] in that the roles of m and n are reversed, consistent with the remark following Definition 4.2.

We have found it occasionally useful to use the above integral representations, but predominantly we prefer Mellin-type integrals, and we next state those formulas. The integral transforms ϕ_h , ϕ_+ and ϕ_- are realized as convolutions on the group $(\mathbb{R}^+, \frac{dx}{x})$ and therefore their Mellin transforms can be easily computed.

Proposition 4.8. *The integral transforms ϕ_h and ϕ_{\pm} have the alternative formulas*

$$(4.37) \quad \phi_h(k) = \frac{1}{2\pi i} \int_{(1)} \frac{2^{s-1} \Gamma\left(\frac{s+k-1}{2}\right)}{\Gamma\left(\frac{k+1-s}{2}\right)} \tilde{\phi}(s+1) (4\pi\sqrt{mn})^{-s} ds,$$

and

$$(4.38) \quad \phi_{\pm}(t) = \frac{1}{2\pi i} \int_{(2)} h_{\pm}(s, t) \tilde{\phi}(s+1) (4\pi\sqrt{mn})^{-s} ds,$$

where

$$h_{\pm}(s, t) = \begin{cases} \frac{1}{\pi} 2^{s-1} \cos(\pi s/2) \Gamma(\frac{s}{2} + it) \Gamma(\frac{s}{2} - it), & \pm = + \\ \frac{1}{\pi} 2^{s-1} \cosh(\pi t) \Gamma(\frac{s}{2} + it) \Gamma(\frac{s}{2} - it), & \pm = -. \end{cases}$$

Proof. By Mellin inversion, we have

$$(4.39) \quad \phi_h(k) = \frac{1}{2\pi i} \int_{(1)} \mathcal{M}(J_{k-1} * (x \cdot \phi), s) (4\pi\sqrt{mn})^{-s} ds.$$

The Mellin transform satisfies the property $\mathcal{M}(f * g, s) = \mathcal{M}(f, s)\mathcal{M}(g, s)$. The Mellin transform of the J -Bessel function (see [11, 6.8 (1)]) is given as

$$(4.40) \quad \int_0^\infty J_\nu(x) x^s \frac{dx}{x} = \frac{2^{s-1} \Gamma(\frac{s+\nu}{2})}{\Gamma(\frac{\nu-s}{2} + 1)}.$$

Also note that $\widetilde{x\phi}(s) = \widetilde{\phi}(s+1)$. Therefore (4.39) can be recast as (4.37), as desired.

For ϕ_- we have by [11] §6.8 (26) that

$$(4.41) \quad \int_0^\infty K_{2it}(x) x^s \frac{dx}{x} = 2^{s-2} \Gamma(\frac{s}{2} + it) \Gamma(\frac{s}{2} - it),$$

and therefore we obtain the minus case of (4.38).

The plus case follows from using (4.40), the reflection formula for the gamma function, and the addition formulas for sin. \square

4.6. Spectral large sieve. Quoting from [3], we have if u_j (g , respectively) is a L^2 -normalized cuspidal Hecke-Maass (holomorphic, resp.) newform of level N with trivial nebentypus, then

$$(4.42) \quad |\nu_{\infty,j}(1)|^2 = N^{-1}(N(1 + |t_j|))^{o(1)}, \quad \text{and} \quad |\nu_{\infty,g}(1)|^2 = N^{-1}(Nk)^{o(1)}.$$

With the normalization (4.28), and assuming \mathfrak{a} is an Atkin-Lehner cusp, the spectral large sieve inequalities give (note that for an Atkin-Lehner cusp \mathfrak{a} , the quantity $\mu(\mathfrak{a})$, in the notation of Deshouillers-Iwaniec [8], is $1/N$):

$$\sum_{|t_j| \leq T} \left| \sum_{m \leq M} a_m \nu_{\mathfrak{a}j}(m) \right|^2 \ll \left(T^2 + \frac{M}{N} \right) (MNT)^\varepsilon \sum_{m \leq M} |a_m|^2,$$

and

$$(4.43) \quad \sum_{\mathfrak{c} \text{ singular}} \int_{|t| \leq T} \left| \sum_{m \leq M} a_m \nu_{\mathfrak{a}\mathfrak{c}}(m, \frac{1}{2} + it) \right|^2 dt \ll \left(T^2 + \frac{M}{N} \right) (MNT)^\varepsilon \sum_{m \leq M} |a_m|^2,$$

and

$$\sum_{k \leq T} \sum_{g \in H_k(N)} \left| \sum_{m \leq M} a_m \nu_{\mathfrak{a}g}(m) \right|^2 \ll \left(T^2 + \frac{M}{N} \right) (MNT)^\varepsilon \sum_{m \leq M} |a_m|^2.$$

4.7. Newforms and oldforms. Atkin and Lehner showed the orthogonal decomposition

$$S_\kappa(N) = \bigoplus_{LM=N} \bigoplus_{f \in H_\kappa^*(M)} S_\kappa(L; f),$$

where $S_\kappa(L; f)$ is the span of forms $f|_\ell$, with $\ell \mid L$, where

$$(4.44) \quad f|_\ell(z) = \ell^{\kappa/2} f(\ell z).$$

Their proof works with virtually no changes to cover the case of Maass forms (which have weight 0, in our context). For the rest of this section, we focus on the Maass case, but with a general weight κ (in order to most easily translate the results to the holomorphic case).

The formula (4.44) means that (let us agree to drop the subscript ∞ when working with the Fourier expansion at ∞)

$$(4.45) \quad \nu_{f|_\ell}(n) = \ell^{1/2} \nu_f(n/\ell).$$

Blomer and Milićević have shown in [6, Section 6] that there exists a basis of $S_\kappa(L; f)$ of the following type. Let f^* denote a newform of level $M|N$, normalized as a *level N* form, by $\langle f^*, f^* \rangle_N = 1$, where

$$(4.46) \quad \langle g_1, g_2 \rangle_N = \int_{\Gamma_0(N) \backslash \mathbb{H}} g_1(z) \overline{g_2(z)} y^\kappa \frac{dx dy}{y^2}.$$

Then there exists an orthonormal basis for $S_\kappa(L; f)$ of the form $g_m = \sum_{\ell|L} c_{\ell,m} f^*|_\ell$, where $c_{\ell,m} \ll N^\varepsilon$. We have (4.42), and for an Atkin-Lehner cusp \mathfrak{a} , we have $|\nu_{\mathfrak{a},f^*}(1)| = |\nu_{\infty,f^*}(1)|$, by (4.11).

The main property we need is the following.

Lemma 4.9. *Suppose \mathfrak{a} is an Atkin-Lehner cusp of $\Gamma_0(N)$, and f^* is a newform of level M with $LM = N$. Then the set of lists of Fourier coefficients $\{(\nu_{\mathfrak{a},f^*|_\ell}(n))_{n \in \mathbb{N}} : \ell|L\}$ is, up to signs, the same as the set of lists of Fourier coefficients $\{(\nu_{\infty,f^*|_\ell}(n))_{n \in \mathbb{N}} : \ell|L\}$.*

Proof. The key fact in the proof is the following. Suppose that p is prime, $p|N$, $p^\alpha||N$, $p^\beta||L$ (so $p^{\alpha-\beta}||M$), and $p^\gamma||\ell$. Let W_{p^α} be the Atkin-Lehner involution for $\Gamma_0(N)$ for the prime p . Then

$$(4.47) \quad (f^*|_\ell)|_{W_{p^\alpha}} = \pm f^*|_{\ell'},$$

where ℓ' is defined by $\ell = p^\gamma h$, (so $(h, p) = 1$), and $\ell' = p^{\beta-\gamma} h$. Note that the map $\ell \rightarrow \ell'$ permutes the divisors of L , and is an involution. Taking (4.47) for granted for a moment, we may complete the proof of Lemma 4.9, by noting that the Fourier coefficients of $f^*|_\ell$ at an Atkin-Lehner cusp \mathfrak{a} are equal to the Fourier coefficients of $(f^*|_\ell)|_{W_D}$ for some Atkin-Lehner involution with $D|N$, which is a composition of W_{p^α} 's. The lemma follows from repeated usage of (4.47).

Now we prove (4.47). First suppose $\gamma \leq \beta/2$, and let $\ell' = p^{\beta-2\gamma}\ell = p^{\beta-\gamma}h$. Then by [1, Lemma 26],

$$(4.48) \quad (f^*|_\ell)|_{W_{p^\alpha}} = (f^*|_{W'_{p^{\alpha-\beta}}})|_{\ell'} = \eta_p(f^*) f^*|_{\ell'},$$

where $W'_{p^{\alpha-\beta}}$ is the Atkin-Lehner involution on $\Gamma_0(M)$ and $\eta_p(f^*)$ is its eigenvalue (technically, they worked with holomorphic forms but their proof works equally well for Maass forms). This proves the claim under the condition $\gamma \leq \beta/2$. If $\gamma > \beta/2$, then one may reverse the roles of ℓ and ℓ' and apply W_{p^α} to both sides of (4.48) to give the result. \square

It is crucial for our later purposes to bound the Hecke eigenvalues of newforms at primes dividing the level. Let f^* be a newform (Maass or holomorphic) of level M as above. If $p|M$, then

$$(4.49) \quad |\lambda_{f^*}(p)| = p^{-1/2},$$

for which see [25, Theorem 3 (iii)] or [1, Theorem 3 (iii)] (the proofs carry over to Maass forms with virtually no changes).

4.8. Fourier coefficients of Eisenstein Series. In Section 11.5 we encounter a Dirichlet series of the form $\sum_{m,n=1}^\infty \nu_{\mathfrak{ac}}(mn, \frac{1}{2} + it) m^{-s} n^{-w}$, where \mathfrak{a} is an Atkin-Lehner cusp and \mathfrak{c} is an arbitrary cusp of $\Gamma_0(N)$. For that purpose, we need to calculate $\nu_{\mathfrak{ac}}(n, \frac{1}{2} + it)$ explicitly for any such cusp pair. The Fourier expansion of the Eisenstein series with general cusps and arbitrary level is surprisingly difficult to find in the literature, so we hope that these formulas shall be useful in other contexts. Huxley [16] has calculated the scattering matrix (which

only involves the constant terms in the Fourier expansion) for a general congruence subgroup, using an alternative basis of Eisenstein series associated to pairs of Dirichlet characters, as opposed to cusps.

4.8.1. *Cusps.* First we write down representatives from the set of Γ -equivalency classes of cusps. An explicit parametrization was given in [8], however it is more convenient for us to assume that each cusp is of the form $1/w$ where now w does not necessarily divide N . For this reason, we include the elementary proof.

Proposition 4.10. *Every cusp of $\Gamma_0(N)$ is equivalent to one of the form $\mathfrak{b} = 1/w$ with $1 \leq w \leq N$. Two cusps of the form $1/w$ and $1/v$ with $1 \leq v, w \leq N$ are equivalent to each other if and only if*

$$(4.50) \quad (v, N) = (w, N), \quad \text{and} \quad \frac{v}{(v, N)} \equiv \frac{w}{(w, N)} \pmod{((w, N), \frac{N}{(w, N)})}.$$

A cusp of the form p/q is equivalent to one of the form $1/w$ with $w \equiv p'q \pmod{N}$ where $p' \equiv p \pmod{(q, N)}$ and $(p', N) = 1$.

Proof. Let $\mathfrak{b} = p/q$ be a cusp. We may take $(p, q) = 1$. Using Bezout's lemma choose $a, b \in \mathbb{Z}$ such that $ap + bq = 1$, and $(a, N) = 1$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. This ensures that $\gamma \cdot \mathfrak{b}$ is a rational number with numerator equal to 1. Replacing c by $c + aNt$ and d with $d + bNt$, we have

$$\begin{pmatrix} a & b \\ c + aNt & d + bNt \end{pmatrix} \cdot \mathfrak{b} = \frac{ap + bq}{(c + aNt)p + (d + bNt)q} = \frac{1}{cp + dq + Nt}.$$

Hence the denominator may be chosen to lie in the interval $[1, N]$. Further note that the denominator is congruent to dq modulo N . From

$$d \equiv \bar{a} \pmod{N} \quad \text{and} \quad a \equiv \bar{p} \pmod{q},$$

we deduce that $d \equiv p \pmod{(N, q)}$. Thus we get the last statement in the proposition.

We have established that any cusp is equivalent to one of the form $1/w$ with $1 \leq w \leq N$. Now let $1/w$ and $1/v$ be two such cusps, and assume that they are equivalent. Elements of the group Γ send relatively prime integer pairs to other such pairs, so if

$$\begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \cdot \frac{1}{w} = \frac{1}{v},$$

then switching the signs on a, b, c, d if necessary, we have

$$(4.51) \quad a + bw = 1 \quad \text{and} \quad Nc + dw = v.$$

The latter equation implies $(v, N) = (dw, N)$. Since $(d, N) = 1$, we get that

$$(4.52) \quad (N, w) = (N, v).$$

The first equation in (4.51) implies $a \equiv 1 \pmod{w}$, which means that $a \equiv 1 \pmod{(N, w)}$. Since the matrix has determinant 1, then $ad \equiv 1 \pmod{N}$, and hence $ad \equiv 1 \pmod{(N, w)}$. Therefore

$$(4.53) \quad d \equiv 1 \pmod{(N, w)}.$$

The second equation in (4.51) gives that $dw \equiv v \pmod{N}$, equivalently

$$d \frac{w}{(w, N)} \equiv \frac{v}{(v, N)} \pmod{\frac{N}{(N, w)}}.$$

Then (4.53) implies

$$(4.54) \quad \frac{w}{(w, N)} \equiv \frac{v}{(v, N)} \pmod{((w, N), \frac{N}{(w, N)})}.$$

Thus we have shown if $1/w$ and $1/v$ are equivalent, and $1 \leq w, v \leq N$, then (4.50) holds.

Now suppose that w, v satisfy (4.50). Let

$$v' = \frac{v}{(N, v)}, \quad w' = \frac{w}{(N, w)}, \quad N' = \frac{N}{(N, w)}.$$

Then $v' \equiv w' \pmod{(N', (N, w))}$, and $(wv', N') = (w, N') = ((N, w), N')$. Therefore there exist $b, c \in \mathbb{Z}$ so that $v' - w' = bwv' + cN'$. That is, $v - w = bwv + cN$. Define $a = 1 - bw$, $d = 1 + bv$, and let $\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix}$, where one may check that γ has determinant 1, so $\gamma \in \Gamma_0(N)$. Finally, one may directly verify that $\gamma \cdot \frac{1}{w} = \frac{1}{v}$. \square

In the notation of Proposition 4.10, call $(w, N) = f$, and $w = uf$.

Corollary 4.11. *A choice of representatives for the set of inequivalent cusps of $\Gamma_0(N)$ is given by*

$$\left\{ \frac{1}{uf} : f|N, u \in (\mathbb{Z}/(f, N/f)\mathbb{Z})^* \right\},$$

and where we choose u so that $(u, N) = 1$, after adding a suitable multiple of $(f, N/f)$.

The cusp 0 is equivalent to $1/1$ and ∞ is equivalent to $1/N$, furthermore $1/uf$ is equivalent to the cusp u/f .

Remark. It is not true that cusps of the form u/f and $1/uf$ are always equivalent, even if $(u, f) = 1$. For that to be true one further requires that $(u, N) = 1$. For example let $N = 72$, and $f = 3$. We have $\frac{2}{3} \not\sim_{\Gamma} \frac{1}{6}$, however it is true that $\frac{2}{3} \sim_{\Gamma} \frac{5}{3} \sim_{\Gamma} \frac{1}{15}$.

For an arbitrary cusp \mathfrak{c} that is not necessarily an Atkin-Lehner cusp, we pick a convenient scaling matrix. First we need to compute the stabilizers of various cusps.

Proposition 4.12. *Let $\mathfrak{c} = 1/w$ be a cusp of $\Gamma = \Gamma_0(N)$, and set*

$$(4.55) \quad N = (N, w)N' \quad w = (N, w)w' = (N', w)w'', \quad N' = (N', w)N''.$$

The stabilizer of $1/w$ is given as

$$(4.56) \quad \Gamma_{1/w} = \left\{ \pm \begin{pmatrix} 1 - w''N't & N''t \\ -w'w''Nt & 1 + w''N't \end{pmatrix} : t \in \mathbb{Z} \right\},$$

and one may choose the scaling matrix as

$$(4.57) \quad \sigma_{\mathfrak{c}} = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \begin{pmatrix} \sqrt{N''} & 0 \\ 0 & 1/\sqrt{N''} \end{pmatrix}.$$

Remark. This is not the choice of scaling matrix we made in Section 4.3 for an Atkin-Lehner cusp. When computing the Fourier coefficients of $E_{\mathfrak{c}}(\sigma_{\mathfrak{a}}z, u)$, with $\mathfrak{a} = 1/r$ Atkin-Lehner and \mathfrak{c} arbitrary, we will choose (4.57) for $\sigma_{\mathfrak{c}}$, and (4.10) for $\sigma_{\mathfrak{a}}$.

Proof. Taking $\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$ so that $\gamma \cdot \frac{1}{w} = \frac{1}{w}$ means that $a + bw = \pm 1$, and $cN + dw = \pm w$ (with the same choice of \pm sign). Consider the $+$ sign case, the $-$ sign following by symmetry. The former equation determines a in terms of b , and the latter is equivalent to $cN' + dw' = w'$. Since $(N', w') = 1$, we have $w'|c$, so define $c = w'r$. Then $d = 1 - rN'$. Finally, the condition that γ has determinant 1 means $bw = -rN'$, which

means $b = N''t$ (say) and $r = -w''t$, giving that γ takes the form as stated in (4.56). One may conversely check that any matrix of the form stated in (4.56) stabilizes $1/w$.

To show the final statement, one easily calculates that for $\gamma \in \Gamma_{1/w}$, as in (4.56), we have $\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}^{-1} \gamma \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} = \pm \begin{pmatrix} 1 & N''t \\ 0 & 1 \end{pmatrix}$. Thus $\sigma_{\mathfrak{c}}^{-1} \gamma \sigma_{\mathfrak{c}} = \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, so (4.1) is satisfied. \square

4.8.2. Eisenstein Series Fourier Coefficients. We compute the Fourier coefficients of $E_{\mathfrak{c}}(z, u)$ at an Atkin-Lehner cusp $\mathfrak{a} = 1/r$ with $rs = N$, $(r, s) = 1$.

Lemma 4.13. *Let $\mathfrak{c} = 1/w$ be any cusp of $\Gamma = \Gamma_0(N)$ and $\mathfrak{a} = 1/r$ an Atkin-Lehner cusp. Let the scaling matrices be as in (4.10) and (4.57). Then*

$$(4.58) \quad \sigma_{\mathfrak{c}}^{-1} \Gamma \sigma_{\mathfrak{a}} = \left\{ \begin{pmatrix} \frac{A}{N''} \sqrt{N''s} & B/\sqrt{N''s} \\ C\sqrt{N''s} & \frac{D}{s} \sqrt{N''s} \end{pmatrix} : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \begin{matrix} C \equiv -wA \pmod{r} \\ D \equiv -wB \pmod{s} \end{matrix} \right\}.$$

Proof. Let us call $\tau_{\mathfrak{c}} = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}$, and $\nu_{\mathfrak{c}} = \begin{pmatrix} \sqrt{N''} & 0 \\ 0 & 1/\sqrt{N''} \end{pmatrix}$, so $\sigma_{\mathfrak{c}} = \tau_{\mathfrak{c}} \nu_{\mathfrak{c}}$. Let $\tau_r \nu_s$ denote the decomposition of the scaling matrix in (4.10). Take $\begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma$ and compute

$$(4.59) \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \tau_{\mathfrak{c}}^{-1} \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \tau_r = \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix} \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \begin{pmatrix} 1 & \frac{s\bar{s}-1}{r} \\ r & s\bar{s} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Considering the product modulo r gives

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} a & * \\ -aw & * \end{pmatrix} \pmod{r},$$

and hence $C \equiv -wA \pmod{r}$. Reducing modulo s , we obtain

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & \bar{r} \\ r & 0 \end{pmatrix} \equiv \begin{pmatrix} * & a\bar{r} \\ * & -wa\bar{r} \end{pmatrix} \pmod{s},$$

so $D \equiv -wB \pmod{s}$.

Now we check that given $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfying the conditions in (4.58), then it is covered by the products of the form in (4.59). For that purpose we compute

$$\tau_{\mathfrak{c}} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tau_r^{-1} = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} s\bar{s} & \frac{1-s\bar{s}}{r} \\ -r & 1 \end{pmatrix}.$$

Modulo r , the lower left entry of this product is congruent to $wA + C \equiv 0 \pmod{r}$, and also congruent to $-Brw - Dr = -r(Bw + D) \equiv 0 \pmod{s}$. This implies that the lower left entry is divisible by N . \square

We next need to work out representatives for $\Gamma_{\infty} \backslash \sigma_{\mathfrak{c}}^{-1} \Gamma \sigma_{\mathfrak{a}} / \Gamma_{\infty}$. Note that the action of Γ_{∞} on both the left and right does not affect the congruences linking A to C and B to D , which follows from $w^2 N'' \equiv 0 \pmod{N}$ (observing $w^2 N'' = w'' w' N$). We need to find the set of pairs C, D with $C > 0$, $(C, D) = 1$, $D \pmod{sC}$, for which there exist integers A, B with $AD - BC = 1$ and so that $C \equiv -wA \pmod{r}$ and $D \equiv -wB \pmod{s}$.

Before stating the result, we develop some notation. Suppose that A, B, C, D are as in the right hand side of (4.58). Write

$$(4.60) \quad f = (w, N), \quad f_r = (w, r), \quad f_s = (w, s),$$

and note $f = f_r f_s$. From $(A, C) = 1$ and $C \equiv -wA \pmod{r}$, we derive $(C, r) = (w, r) = f_r$. Similarly, $(D, s) = (w, s) = f_s$. Write

$$(4.61) \quad C = f_r C', \quad r = f_r r', \quad D = f_s D', \quad s = f_s s',$$

where $(C', r') = 1 = (D', s')$. Then we have

$$(4.62) \quad \begin{aligned} C &\equiv -wA \pmod{r} \\ D &\equiv -wB \pmod{s} \end{aligned} \iff \begin{aligned} A &\equiv -\overline{(w/f_r)}C' \pmod{r'} \\ B &\equiv -\overline{(w/f_s)}D' \pmod{s'} \end{aligned}.$$

The equivalence of congruences in (4.62) only uses the assumptions $(A, C) = (B, D) = 1$.

From $AD - BC = 1$, we have $D \equiv \overline{A} \pmod{|C|}$, which combined with the right hand side of (4.62) gives

$$D \equiv -\overline{C'} \frac{w}{f_r} \pmod{(f_r, r')},$$

using $(C, r') = (f_r C', r') = (f_r, r')$. Similarly, we have $B \equiv -\overline{C} \pmod{|D|}$, so $D' \equiv \overline{C} \frac{w}{f_s} \pmod{(f_s, s')}$.

Lemma 4.14. *With notation as in Lemma 4.13 and its following discussion, we have the disjoint union*

$$(4.63) \quad \Gamma_\infty \backslash \sigma_{\mathfrak{c}}^{-1} \Gamma \sigma_{\mathfrak{a}} / \Gamma_\infty \cong \delta_{\mathfrak{ca}} \Omega_\infty \cup \left\{ \begin{pmatrix} * & * \\ C\sqrt{N''s} & \frac{D}{s}\sqrt{N''s} \end{pmatrix} : \begin{aligned} (C, r) &= (w, r) & D &\equiv -\overline{C'} \frac{w}{f_r} \pmod{(f_r, r')} \\ (D, s) &= (w, s) & D' &\equiv \overline{C} \frac{w}{f_s} \pmod{(f_s, s')} \end{aligned} \right\},$$

where if \mathfrak{c} is equivalent to \mathfrak{a} then $\Omega_\infty = \Gamma_\infty \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \Gamma_\infty$, and where in addition we have the restrictions $C > 0$, $(C, D) = 1$ and D runs over representatives \pmod{sC} .

Proof. The discussion preceding the statment of the lemma shows that any matrix given on the right hand side of (4.58) with $C > 0$ gives rise to a double coset of the claimed form. It suffices to show that given integers C, D as in the second line of (4.63), we may find A, B so that $AD - BC = 1$ and satisfying the congruences in (4.62).

From $(C, D) = 1$, we may choose A_0, B_0 so that $A_0 D - B_0 C = 1$. From $A_0 \equiv \overline{D} \pmod{C}$, and the congruence on D given in (4.63), we have

$$A_0 \equiv -\overline{(w/f_r)}C' \pmod{(f_r, r')}.$$

Hence, there exist integers x, y so that $A_0 + C'\overline{(w/f_r)} = f_r x + r' y$. A similar argument with B_0 gives

$$B_0 \equiv -\overline{(w/f_s)}D' \pmod{(f_s, s')},$$

and so there exist integers u, v so that $B_0 + D'\overline{(w/f_s)} = f_s u + s' v$.

We next want to find $n \in \mathbb{Z}$ so that $A = A_0 + nC = A_0 + nC'f_r$ and $B = B_0 + nD = B_0 + nD'f_s$ satisfies the right hand side of (4.62). Gathering the above formulas, we have

$$\begin{aligned} A &= A_0 + nC = -\overline{(w/f_r)}C' + f_r(x + nC') + r'y \\ B &= B_0 + nD = -\overline{(w/f_s)}D' + f_s(u + nD') + s'v. \end{aligned}$$

We may choose n so that $n \equiv -\overline{C'}x \pmod{r'}$ and $n \equiv -\overline{D'}u \pmod{s'}$, since $(C', r') = 1 = (D', s')$, and by the Chinese remainder theorem. With this choice of n , then A and B satisfy the congruences on the right hand side of (4.62). \square

Finally we can evaluate $\phi_{\mathfrak{ac}}(n, u)$. First, note that the congruence $D \equiv -\overline{C'} \frac{w}{f_r} \pmod{(f_r, r')}$ is equivalent to $D' \equiv -\overline{C'} f_s \frac{w}{f_r} \pmod{(f_r, r')}$, and that we can write $w = f_r f_s w'$, giving now $D' \equiv -\overline{C'} w' \pmod{(f_r, r')}$. Similarly, the other congruence in (4.63) is equivalent to $D' \equiv \overline{C'} w' \pmod{(f_s, s')}$.

Putting everything together, we have

$$(4.64) \quad \phi_{\text{ac}}(n, u) = \frac{1}{(N'' s f_r^2)^u} \sum_{(C', f_s r')=1} \frac{1}{(C')^{2u}} \sum_{\substack{D' \pmod{s' f_r C'} \\ D' \equiv -\overline{C'} w' \pmod{(f_r, r')}}}^* e\left(\frac{n D'}{s' f_r C'}\right).$$

Now write

$$f_r = f'_r f_0, \quad \text{where} \quad (f_0, r') = 1, \quad \text{and} \quad f'_r | (r')^\infty,$$

and similarly

$$s' = s'_f s_0, \quad \text{where} \quad (s_0, f_s) = 1, \quad \text{and} \quad s'_f | f_s^\infty.$$

Then $(f_r, r') = (f'_r, r')$, and $(f_s, s') = (f_s, s'_f)$, and so

$$\phi_{\text{ac}}(n, u) = \frac{1}{(N'' s f_r^2)^u} \sum_{(C', f_s r')=1} \frac{1}{(C')^{2u}} \sum_{\substack{D' \pmod{s_0 f_0 C' f'_r s'_f} \\ D' \equiv -\overline{C'} w' \pmod{(f'_r, r')}}}^* e\left(\frac{n D'}{s_0 f_0 C' f'_r s'_f}\right).$$

By the Chinese remainder theorem, this factors as

$$\begin{aligned} \phi_{\text{ac}}(n, u) &= \frac{1}{(N'' s f_r^2)^u} \sum_{(C', f_s r')=1} \frac{S(n, 0; C' s_0 f_0)}{(C')^{2u}} \\ &\times \left(\sum_{\substack{D' \pmod{f'_r} \\ D' \equiv -\overline{C'} w' \pmod{(f'_r, r')}}}^* e\left(\frac{n D' \overline{s_0 f_0 C' f'_r}}{f'_r}\right) \right) \left(\sum_{\substack{D' \pmod{s'_f} \\ D' \equiv \overline{C'} w' \pmod{(f_s, s'_f)}}}^* e\left(\frac{n D' \overline{s_0 f_0 C' f'_r}}{s'_f}\right) \right). \end{aligned}$$

For the sum modulo f'_r , note that $(D', f'_r) = 1$ if and only if $(D', (f'_r, r')) = 1$, since $f'_r | r'^\infty$. Therefore, the congruence automatically implies $(D', f'_r) = 1$, and so this condition may be dropped. Then after changing variables, it becomes a linear exponential sum which is easy to evaluate. A similar discussion holds for the modulus s'_f . In this way, we obtain

$$\begin{aligned} \phi_{\text{ac}}(n, u) &= \frac{1}{(N'' s f_r^2)^u} \frac{f'_r}{(f'_r, r')} \delta\left(\frac{f'_r}{(f'_r, r')} | n\right) \frac{s'_f}{(s'_f, f_s)} \delta\left(\frac{s'_f}{(s'_f, f_s)} | n\right) \\ &\sum_{(C', f_s r')=1} \frac{S(n, 0; C' s_0 f_0)}{(C')^{2u}} e\left(\frac{-n w' \overline{s_0 f_0 s'_f C'^2}}{f'_r}\right) e\left(\frac{n w' \overline{s_0 f_0 f'_r C'^2}}{s'_f}\right). \end{aligned}$$

Now write

$$n = \frac{f'_r}{(f'_r, r')} \frac{s'_f}{(s'_f, f_s)} k,$$

and note $(\frac{f'_r}{(f'_r, r')} \frac{s'_f}{(s'_f, f_s)}, C' s_0 f_0) = 1$ giving

$$\begin{aligned} \phi_{\text{ac}}(n, u) &= \frac{S(k, 0; s_0 f_0)}{(N'' s f_r^2)^u} \frac{f'_r}{(f'_r, r')} \frac{s'_f}{(s'_f, f_s)} \\ &\sum_{(C', f_s r')=1} \frac{S(k, 0; C')}{(C')^{2u}} e\left(\frac{-k w' \overline{s_0 f_0 (s'_f, f_s) C'^2}}{(f'_r, r')}\right) e\left(\frac{k w' \overline{s_0 f_0 (f'_r, r') C'^2}}{(s'_f, f_s)}\right). \end{aligned}$$

Next we use the evaluation of the Ramanujan sum as a divisor sum, giving

$$\begin{aligned} \phi_{\text{ac}}(n, u) &= \frac{S(k, 0; s_0 f_0)}{(N'' s f_r^2)^u} \frac{f'_r}{(f'_r, r')} \frac{s'_f}{(s'_f, f_s)} \sum_{\substack{d|k \\ (d, f_s r')=1}} d^{1-2u} \\ &\quad \sum_{(C', f_s r')=1} \frac{\mu(C')}{(C')^{2u}} e\left(\frac{-k w' s_0 f_0 (s'_f, f_s) d^2 C'^2}{(f'_r, r')}\right) e\left(\frac{k w' s_0 f_0 (f'_r, r') d^2 C'^2}{(s'_f, f_s)}\right). \end{aligned}$$

Our next goal is to change the additive character into multiplicative ones. This requires placing the fractions in lowest terms.

Tracing back the definitions, we may check that $(w', (f'_r, r')) = 1$, since this is equivalent to $(w', r') = 1$, which is true by the definitions of w' and r' . Similarly, $(w', (s'_f, f_s)) = 1$. Next write $k = k_r k_s \ell$, where

$$k_r = (k, (f'_r, r')), \quad k_s = (k, (s'_f, f_s)).$$

After this, we may convert to Dirichlet characters (see [21, (3.11)]). In all, we obtain

$$\begin{aligned} (4.65) \quad \phi_{\text{ac}}(n, u) &= \frac{S(\ell, 0; s_0 f_0)}{(N'' s f_r^2)^u} \frac{f'_r}{(f'_r, r')} \frac{s'_f}{(s'_f, f_s)} \sum_{\substack{d|k \\ (d, f_s r')=1}} d^{1-2u} \frac{1}{\varphi(\frac{(f'_r, r')}{k_r})} \frac{1}{\varphi(\frac{(s'_f, f_s)}{k_s})} \\ &\quad \sum_{\chi \pmod{\frac{(f'_r, r')}{k_r}}} \sum_{\psi \pmod{\frac{(s'_f, f_s)}{k_s}}} \frac{(\chi\psi)(\ell) \tau(\overline{\chi}) \tau(\overline{\psi})}{L(2u, \chi^2 \psi^2 \chi_0)} (\chi\psi)(s_0 f_0 d^2 w') \chi(-k_s \overline{(s'_f, f_s)}) \psi(k_r \overline{(f'_r, r')}), \end{aligned}$$

where χ_0 is the principal character modulo $f_s r'$. For later calculations, it will be useful to notice that the condition $(\frac{k}{k_r}, \frac{(f'_r, r')}{k_r}) = 1$ (and similarly in the s -aspect) is automatic from the presence of $(\chi\psi)(\ell)$. Moreover, we have $(f'_r, r') = (f_r, \frac{r}{f_r})$, and similarly $(s'_f, f_s) = (f_s, \frac{s}{f_s})$, and so $(f'_r, r')(s'_f, f_s) = (f, \frac{N}{f})$. Note the condition $d|k$ together with $(d, f_s r') = 1$ implies $d|\ell$.

5. INERT FUNCTIONS

5.1. Basic Definition. We begin with a definition of a useful class of functions.

Definition 5.1. Suppose that for each $q \geq 1$, $\{w_T\}_{T \in \mathcal{F}_q}$ is a collection of smooth functions on \mathbb{R}_+^d with support on a product of dyadic intervals. Further suppose that $X = X_q \geq 1$ is a non-decreasing function of q .

We say that the family of functions $\{w_T : q \geq 1, T \in \mathcal{F}_q\}$ is X -inert if for all $j_1, \dots, j_d \geq 0$, there exists a constant $C = C_{\mathcal{F}}(j_1, \dots, j_d) \in \mathbb{R}_+$ so that for $T \in \mathcal{F}_q$,

$$(5.1) \quad X_q^{-j_1 - \dots - j_d} |w_T^{(j_1, \dots, j_d)}(x_1, \dots, x_d)| \leq \frac{C_{\mathcal{F}}(j_1, \dots, j_d)}{|x_1|^{j_1} \dots |x_d|^{j_d}}.$$

By abuse of language, we will sometimes say that w_T is inert, or that it is uniformly inert, if the family is clear from context. Also we will not specify the filtration $\mathcal{F} = \cup_q \mathcal{F}_q$ if the family \mathcal{F} is 1-inert.

The purpose of this definition is to encode natural conditions on a weight function that lets us separate variables efficiently. For instance, if w_T satisfies (5.1), then by Mellin inversion,

$$(5.2) \quad w_T(x_1, \dots, x_d) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widetilde{w}_T(it_1, \dots, it_d) x_1^{-it_1} \dots x_d^{-it_d} dt_1 \dots dt_d,$$

where

$$\widetilde{w}_T(s_1, \dots, s_d) = \int_{(0, \infty)^d} w_T(x_1, \dots, x_d) x_1^{s_1} \dots x_d^{s_d} \frac{dx_1}{x_1} \dots \frac{dx_d}{x_d}.$$

Integrating by parts shows

$$\widetilde{w}_T(s_1, \dots, s_d) = \prod_{a=1}^d \prod_{b=0}^{j_a-1} \frac{1}{(s_a + b)} \int_{(0, \infty)^d} w_T^{(j_1, \dots, j_d)}(x_1, \dots, x_d) x_1^{s_1+j_1} \dots x_d^{s_d+j_d} \frac{dx_1}{x_1} \dots \frac{dx_d}{x_d}.$$

Therefore, by (5.1), we have

$$|\widetilde{w}_T(it_1, \dots, it_d)| \leq \left(\frac{X}{|t_1|}\right)^{j_1} \dots \left(\frac{X}{|t_d|}\right)^{j_d} C(j_1, \dots, j_d) (\log 2)^d.$$

If $|t_i| \geq X$, then we take j_i as unspecified, while if $|t_i| < X$, we choose $j_i = 0$. In this way, we obtain

$$|\widetilde{w}_T(it_1, \dots, it_d)| \leq \left(1 + \frac{|t_1|}{X}\right)^{-j_1} \dots \left(1 + \frac{|t_d|}{X}\right)^{-j_d} C'(j_1, \dots, j_d),$$

where C' is some other sequence depending only on C . Our interpretation of this estimate combined with (5.2) is that w_T can have its variables separated “at cost” X^d , meaning that each integral in (5.2) has essential length $\ll X$.

The main gain with the definition of inert is that it lets us perform this separation of variables at a later stage, while keeping track of the “cost” of the separation. A more naive approach would separate the variables at the beginning, but this has a downside in that the notation becomes immediately cumbersome. In addition, some intermediate steps such as stationary phase lead to new weight functions that no longer have separated variables; it would then be necessary to re-separate the variables. The notion of an inert function is thus more robust, because it is preserved by stationary phase (see Section 5.4).

In our desired applications, for each $q \geq 1$, our family of weight functions $\{w_T\}_{T \in \mathcal{F}_q}$ will be indexed by tuples T of the form $T = (M_1, M_2, N_1, N_2, N_3, C, a, \dots)$, as well as some other parameters that arise as dual variables after Poisson summation. Furthermore, the relevant values of X will be $c(\varepsilon)q^\varepsilon$ for some constant $c(\varepsilon)$.

One may easily check that if $\{w_T\}_{T \in \mathcal{F}}$ and $\{w_{T'}\}_{T' \in \mathcal{F}'}$ are X -inert families, then so is $\{w_T \cdot w_{T'}\}_{(T, T') \in \mathcal{F}_q \times \mathcal{F}'_q, q \geq 1}$, where the list of constants could be written explicitly in terms of those for \mathcal{F} and \mathcal{F}' . For instance, in the one-variable case, we have

$$|X^{-j} \frac{d^j}{dx^j} w_T(x) w_{T'}(x)| \leq |x|^{-j} \sum_{k=0}^j \binom{k}{j} C(j) C'(k-j).$$

Example. Let $w(x_1, \dots, x_d)$ be a smooth function that is supported on $[1, 2]^d$ and define, for quantities M_1, \dots, M_d , the function

$$w_{M_1, \dots, M_d}(x_1, \dots, x_d) = w\left(\frac{x_1}{M_1}, \dots, \frac{x_d}{M_d}\right).$$

Then for any collection \mathcal{F} of tuples $(M_1, \dots, M_d) \in \mathbb{R}_{>0}^d$, the family \mathcal{F} is 1-inert.

5.2. Fourier transforms. Inert functions behave regularly under the Fourier transform. Suppose that $w_T(x_1, \dots, x_d)$ is X -inert, and let

$$\widehat{w}_T(t_1, x_2, \dots, x_d) = \int_{-\infty}^{\infty} w_T(x_1, \dots, x_d) e(-x_1 t_1) dx_1$$

denote its Fourier transform in the x_1 -variable. Suppose that the support of w_T is such that $x_i \asymp X_i$ for each i . Now \widehat{w}_T is not compactly-supported in t_1 , so it will not be inert. However, if we let $W_{T,Y_1}(t_1, x_2, \dots, x_d) = w_{Y_1}(t_1) \widehat{w}_T(t_1, x_2, \dots, x_d)$ where $\{w_{Y_1} : Y_1 > 0\}$ is a 1-inert family, supported on $t_1 \asymp Y_1$ (or $-t_1 \asymp Y_1$) then $X_1^{-1} W_{T,Y_1}$ forms an X -inert family, as we now discuss. We have

$$\begin{aligned} \frac{d^{j_1}}{dt_1^{j_1}} X_1^{-1} \widehat{w}_T(t_1, x_2, \dots, x_d) &= \frac{d^{j_1}}{dt_1^{j_1}} \int_{-\infty}^{\infty} w_T(X_1 x_1, \dots, x_d) e(-x_1 X_1 t_1) dx_1 \\ &= \int_{-\infty}^{\infty} w_T(X_1 x_1, \dots, x_d) (-2\pi i x_1 X_1)^{j_1} e(-x_1 X_1 t_1) dx_1 \ll X_1^{j_1}, \end{aligned}$$

trivially. We want to show this is $\ll |t_1|^{-j_1} X^{j_1}$. If $|t_1| \ll \frac{X}{X_1}$, then this is satisfactory, so assume $|t_1| \gg \frac{X}{X_1}$. Integrating by parts j_1 times inside the integral gives

$$\frac{d^{j_1} X_1^{-1} \widehat{w}_T(t_1, x_2, \dots, x_d)}{dt_1^{j_1}} = \int_{-\infty}^{\infty} \frac{d^{j_1}}{dx_1^{j_1}} \left[w_T(X_1 x_1, \dots, x_d) x_1^{j_1} \right] e(-x_1 X_1 t_1) \frac{dx_1}{(-t_1)^{j_1}} \ll \frac{X^{j_1}}{|t_1|^{j_1}},$$

since the j -th derivative of the expression in square brackets is $\ll X^j$, and is supported on $x_1 \asymp 1$. By a slight generalization of this to allow derivatives with respect to x_2, \dots, x_d , we see that $X_1^{-1} \widehat{w}_T(t_1, x_2, \dots, x_d)$ satisfies the desired derivative bound that an inert function is required to have. The missing property is that it is not dyadically supported in t_1 (with $t_1 > 0$). However, multiplication by a dyadic function does not appreciably affect the derivative bounds, so that the desired claim on $X_1^{-1} W_{T,Y_1}$ is proven.

Moreover, we have that

$$X_1^{-1} W_{T,Y_1}(t_1, x_2, \dots, x_d) \ll \left(1 + \frac{|t_1| X_1}{X}\right)^{-A} \asymp \left(1 + \frac{Y_1 X_1}{X}\right)^{-A},$$

so that in practice we may restrict our attention to $Y_1 \ll \frac{X q^\varepsilon}{X_1}$.

The above discussion implies the following proposition.

Proposition 5.2. *Suppose that $\{w_T : T \in \mathcal{F}\}$ is a family of X -inert functions such that x_1 is supported on $x_1 \asymp X_1$, and $\{w_{Y_1} : Y_1 \in \pm(0, \infty)\}$ is a 1-inert family of functions with support on $t_1 \asymp Y_1$. Then the family $\{X_1^{-1} w_{Y_1}(t_1) \widehat{w}_T(t_1, x_2, \dots, x_d) : (T, \pm Y_1) \in \mathcal{F} \times (0, \infty)\}$ is X -inert. Furthermore if $Y_1 \gg q^\varepsilon X/X_1$ then for any $A > 0$, we have*

$$X_1^{-1} w_{Y_1}(t_1) \widehat{w}_T(t_1, x_2, \dots, x_d) \ll_A (Y_1 q)^{-A}.$$

5.3. Mellin transforms.

Lemma 5.3. *Let $W_T(x_1, x_2, \dots, x_d)$ be a family of X -inert functions such that x_1 is supported in the dyadic interval $[X_1, 2X_1]$. Let*

$$\widetilde{W}_T(s, x_2, \dots, x_d) = \int_0^\infty W_T(x, x_2, \dots, x_d) x^s \frac{dx}{x}.$$

Then we have $\widetilde{W}_T(s, x_2, \dots, x_d) = X_1^s \omega_T(s, x_2, \dots, x_n)$ where $w_T(s, \cdot)$ is a family of X -inert functions in all the x_i , which is entire in s and has rapid decay for $|\operatorname{Im}(s)| \gg X^{1+\varepsilon}$.

5.4. Stationary phase. Next we synthesize both Lemma 8.1 and Proposition 8.2 of [4] using this language of inert functions, along with some simplified choices of parameters, with the following

Lemma 5.4 ([4]). *Suppose that $w = w_T$ is a family of $X = X_q$ -inert functions, with compact support on $[Z, 2Z]$, so that $w^{(j)}(t) \ll (Z/X)^{-j}$. Also suppose that ϕ is smooth and satisfies $\phi^{(j)}(t) \ll \frac{Y}{Z^j}$ for some $Y \gg X^2 q^\varepsilon$ and all t in the support of w . Let*

$$I = \int_{-\infty}^{\infty} w(t) e^{i\phi(t)} dt.$$

- (1) *If $\phi'(t) \gg \frac{Y}{Z}$ for all t in the support of w , then $I \ll_A q^{-A}$ for A arbitrarily large.*
 (2) *If $\phi''(t) \gg \frac{Y}{Z^2}$ for all t in the support of w , and there exists $t_0 \in \mathbb{R}$ such that $\phi'(t_0) = 0$ (note t_0 is necessarily unique), then*

$$(5.3) \quad I = \frac{e^{i\phi(t_0)}}{\sqrt{\phi''(t_0)}} F_T(t_0) + O(q^{-A}),$$

where F_T is a family of X -inert functions of t_0 (depending on A) supported on $t_0 \asymp Z$.

In case it is useful in other contexts, we mention that statement (1) only requires $Y \gg Xq^\varepsilon$; in our applications in this article we have $1 \ll X \ll q^\varepsilon$, so this has no practical effect.

The part of the conclusion that F_T is a family of X -inert functions of t_0 is not explicitly stated that way in [4], but is implicit in [4, (8.11)]. However, what is required in this paper is a multi-variable version of inertness, which is not directly addressed in [4]. This more general result is the following.

Lemma 5.5. *Suppose w_T is X -inert in t_1, \dots, t_d , and ϕ satisfies*

$$(5.4) \quad \frac{\partial^{a_1+a_2+\dots+a_d}}{\partial t_1^{a_1} \dots \partial t_d^{a_d}} \phi(t_1, t_2, \dots, t_d) \ll \frac{Y}{Z^{a_1}} \frac{X^{a_2+\dots+a_d}}{X_2^{a_2} \dots X_d^{a_d}},$$

for $t_1 \asymp Z$, $t_i \asymp X_i$ for $i = 2, \dots, d$. Assume the conditions in Lemma 5.4 part (2) hold for $t = t_1$ (uniformly in t_2, \dots, t_d), and that t_0 satisfies

$$(5.5) \quad \frac{\partial^{b_2+\dots+b_d}}{\partial t_2^{b_2} \dots \partial t_d^{b_d}} t_0 \ll_{b_2, \dots, b_d} \frac{t_0}{X_2^{b_2} \dots X_d^{b_d}},$$

for $t_0 \asymp Z$ (that is, $\frac{1}{Z}t_0$ is 1-inert). Then F_T is X -inert in t_2, \dots, t_d .

A simple yet common situation is when t_0 is monomial in the other variables, meaning

$$(5.6) \quad t_0 = ct_2^{\alpha_2} \dots t_d^{\alpha_d},$$

where the α_i are fixed real numbers and c is some constant (depending on the tuple T). It is easy to check that if t_0 satisfies (5.6), then it satisfies (5.5). All the applications of Lemma 5.4 in this paper will have the stationary point of the form (5.6).

Lemma 5.6. *Suppose ϕ satisfies (5.4) and the conditions in Lemma 5.4 part (2) hold for $t = t_1$ (uniformly in t_2, \dots, t_d). Then $\frac{1}{Z}t_0$ (with t_0 defined implicitly by $\phi'(t_0, t_2, \dots, t_d) = 0$) is X -inert. In particular, if ϕ satisfies (5.4) with $X = 1$, then the assumption (5.5) may be omitted from the statement of Lemma 5.5.*

The reader may wonder, then, why we have retained the assumption (5.5) in Lemma 5.5. One reason is that in many important cases, it is easy to verify (5.5) directly (e.g. when (5.6) holds). Another reason is that our proof of Lemma 5.6 builds naturally on the proof of Lemma 5.5.

Proof of Lemma 5.5. Suppose w_T is of the form $w_T(t_1, \dots, t_d)$, where T is a tuple of parameters, and similarly ϕ is of the form $\phi(t_1, \dots, t_d)$. Let us consider t_2, \dots, t_d as temporarily fixed, and consider an oscillatory integral in t_1 meeting the conditions of Lemma 5.4, part (2). The bound [4, (8.11)] gives that

$$\frac{d^j}{dy^j} F_T(y) \ll (X/Z)^j \asymp (X/y)^j.$$

However, this estimate views t_2, \dots, t_d as fixed, and does not give bounds on the derivatives of F with respect to t_i with $2 \leq i \leq d$. To go further, we need to recall the origin of $F = F_T$. Namely, we have

$$F(y) = \sum_n p_n(y), \quad p_n(y) = c_n(\phi''(y))^{-n} G_y^{(2n)}(t) \Big|_{t=y},$$

where the sum over n is finite (depending only on the desired value of A in (5.3)), c_n are certain absolute constants, and

$$G_y(t) = w_T(t, t_2, \dots, t_d) e^{iH(t, y, t_2, \dots, t_d)},$$

where (with ϕ'' representing the second derivative in the first variable)

$$H(t, y, t_2, \dots, t_d) = \phi(t, t_2, \dots, t_d) - \phi(y, t_2, \dots, t_d) - \frac{1}{2} \phi''(y, t_2, \dots, t_d) (t - y)^2.$$

Remark. Within the definition of p_n (and hence F), the symbol y is an arbitrary real in the support of w_T . Within (5.3), we then substitute $y = t_0$, where now t_0 is an implicit function of the other variables.

Now we may write p_n more explicitly as a function of y, t_2, \dots, t_d as

$$(5.7) \quad p_n(y, t_2, \dots, t_d) = \left(\frac{1}{\phi''(y, t_2, \dots, t_d)} \right)^n \frac{\partial^{2n}}{\partial t^{2n}} (w_T(t, t_2, \dots, t_d) e^{iH(t, y, t_2, \dots, t_d)}) \Big|_{t=y}.$$

We see that $G_y^{(2n)}(y)$ is a sum of scalar multiples of terms of the form

$$w_T^{(\nu_0)}(y, t_2, \dots, t_d) H^{(\nu_1)}(y, y, t_2, \dots, t_d) \cdots H^{(\nu_\ell)}(y, y, t_2, \dots, t_d),$$

where the superscripts refer to partial differentiation in the first variable with $\nu_0 \geq 0$ and $\nu_1, \dots, \nu_\ell \geq 1$, and where $\nu_0 + \nu_1 + \dots + \nu_\ell = 2n$. Note that $H^{(\nu)}(y, y, t_2, \dots, t_d) = \phi^{(\nu)}(y, t_2, \dots, t_d)$ for $\nu \geq 3$, and vanishes otherwise. We therefore deduce that

$$(5.8) \quad G_y^{(2n)}(y) \ll \max_{\nu_0 + \nu_1 + \dots + \nu_\ell = 2n} \left(\frac{X}{Z} \right)^{\nu_0} \frac{Y^\ell}{Z^{\nu_1 + \dots + \nu_\ell}} \ll \frac{X^{2n} + Y^{2n/3}}{Z^{2n}},$$

seen as follows. Since we may assume $\nu_i \geq 3$ for $i \geq 1$, we have $3\ell \leq \nu_1 + \dots + \nu_\ell = 2n - \nu_0$, whence $X^{\nu_0} Y^\ell \leq (X/Y^{1/3})^{\nu_0} Y^{2n/3}$, which is acceptable for $X \leq Y^{1/3}$. If $X \geq Y^{1/3}$, then we use $Y^\ell X^{\nu_0} \leq X^{3\ell + \nu_0} \leq X^{2n}$.

Let $G_n(y, t_2, \dots, t_d) = G_y^{(2n)}(y, t_2, \dots, t_d)$. A slight generalization of (5.8) shows

$$G_n^{(a_1, a_2, \dots, a_d)}(y, t_2, \dots, t_d) \ll \frac{X^{2n} + Y^{2n/3}}{Z^{2n}} \frac{X^{a_1 + a_2 + \dots + a_d}}{Z^{a_1} X_2^{a_2} \cdots X_d^{a_d}}.$$

The meaning of the superscripts on G_n now mean differentiation with respect to the different variables, viewing y as independent from t_2, \dots, t_d .

Next we examine $\Phi_n(y, \dots, t_d) := (\phi''(y, t_2, \dots, t_d))^{-n}$. We claim

$$(5.9) \quad \Phi_n^{(a_1, a_2, \dots, a_d)}(y, t_2, \dots, t_d) \ll (Z^2/Y)^n \frac{1}{Z^{a_1}} \frac{X^{a_2+\dots+a_d}}{X_2^{a_2} \dots X_d^{a_d}}.$$

For this, we first note that an easy induction argument gives

$$\frac{d^a}{dx^a} \frac{1}{f(x)} = \sum_{j_1+\dots+j_a=a} c_{j_1, \dots, j_a} \frac{(f^{(j_1)}(x)) \dots (f^{(j_a)}(x))}{f(x)^{a+1}},$$

for certain constants c_{j_1, \dots, j_a} . Next we generalize to multiple variables. Let \mathbf{j}_i be a d -tuple of nonnegative integers, and let $\mathbf{a} = (a_1, \dots, a_d)$. Then

$$(5.10) \quad \frac{\partial^{a_1+\dots+a_d}}{\partial y^{a_1} \dots \partial t_d^{a_d}} \frac{1}{f(y, t_2, \dots, t_d)} = \sum_{\mathbf{j}_1+\mathbf{j}_2+\dots+\mathbf{j}_{\mathbf{a}\cdot\mathbf{1}}=\mathbf{a}} c_{\mathbf{j}_1, \dots, \mathbf{j}_{\mathbf{a}\cdot\mathbf{1}}} \frac{f^{(\mathbf{j}_1)} \dots f^{(\mathbf{j}_{\mathbf{a}\cdot\mathbf{1}})}}{f^{\mathbf{a}\cdot\mathbf{1}+1}},$$

where $\mathbf{1}$ is the 1-vector of length d (so $\mathbf{a} \cdot \mathbf{1} = a_1 + \dots + a_d$), and $c_{\mathbf{j}_1, \dots, \mathbf{j}_{\mathbf{a}\cdot\mathbf{1}}}$ are absolute constants. This may be easily verified by induction.

One may show directly from (5.4) that

$$(5.11) \quad \frac{\partial^{b_1+\dots+b_d}}{\partial y^{b_1} \dots \partial t_d^{b_d}} (\phi''(y, t_2, \dots, t_d))^n \ll \left(\frac{Y}{Z^2}\right)^n \frac{1}{Z^{b_1}} \frac{X^{b_2+\dots+b_d}}{X_2^{b_2} \dots X_d^{b_d}}.$$

Combining (5.10) with (5.11), we derive that

$$\Phi_n^{(a_1, \dots, a_d)}(y, t_2, \dots, t_d) \ll \sum_{\mathbf{j}_1+\mathbf{j}_2+\dots+\mathbf{j}_{\mathbf{a}\cdot\mathbf{1}}=\mathbf{a}} \frac{\left(\frac{Y}{Z^2}\right)^{n(\mathbf{a}\cdot\mathbf{1})} \prod_{k=2}^d \left(\frac{X}{X_k}\right)^{\mathbf{j}_1 \cdot \mathbf{e}_k + \dots + \mathbf{j}_{\mathbf{a}\cdot\mathbf{1}} \cdot \mathbf{e}_k}}{Z^{\mathbf{j}_1 \cdot \mathbf{e}_1 + \dots + \mathbf{j}_{\mathbf{a}\cdot\mathbf{1}} \cdot \mathbf{e}_1} \left(\frac{Y}{Z^2}\right)^{n(\mathbf{a}\cdot\mathbf{1}+1)}},$$

where \mathbf{e}_k is the k th standard basis vector. This simplifies to give the claimed (5.9).

Putting the above bounds together, we derive that

$$(5.12) \quad p_n^{(a_1, \dots, a_d)}(y, t_2, \dots, t_d) \ll \left(\left(\frac{X^2}{Y}\right)^n + \left(\frac{1}{Y^{1/3}}\right)^n \right) \frac{X^{a_1+\dots+a_d}}{Z^{a_1} X_2^{a_2} \dots X_d^{a_d}}.$$

Since $Y \gg X^2 q^\varepsilon \gg q^\varepsilon$, this gives an asymptotic expansion in n (as in [4]), and each p_n is X -inert in all variables. Therefore, F is also X -inert in all variables (again, viewing y as an independent variable).

As a final step we need to incorporate the fact that t_0 , which is substituted for y , is not an independent variable but a function of t_2, \dots, t_d . We may derive the shape of a general derivative of F as follows. Let $\mathbf{a} = (a_2, \dots, a_d)$, $\mathbf{j} = (j_2, \dots, j_d)$, $\mathbf{k} = (k_2, \dots, k_d)$, and \mathbf{b}_i be d -tuples of nonnegative integers. We claim

$$(5.13) \quad \frac{\partial^{a_2+\dots+a_d}}{\partial t_2^{a_2} \dots \partial t_d^{a_d}} F(t_0, t_2, \dots, t_d) = \sum_{\mathbf{j}+\mathbf{k} \leq \mathbf{a}} \sum_{N \leq j_2+\dots+j_d} \sum_{\mathbf{b}_1+\dots+\mathbf{b}_N=\mathbf{a}} c_{\mathbf{j}, \mathbf{k}, \mathbf{b}_1, \dots, \mathbf{b}_N} F^{(j_2+\dots+j_d, k_2, \dots, k_d)} t_0^{(\mathbf{b}_1)} \dots t_0^{(\mathbf{b}_N)},$$

where the condition $\mathbf{j} + \mathbf{k} \leq \mathbf{a}$ is interpreted componentwise (so $j_\ell + k_\ell \leq a_\ell$ for all ℓ), and the c_* 's are absolute constants. Moreover, we emphasize that the notation $F^{(j_2+\dots+j_d, k_2, \dots, k_d)}$ here

and below represents partial differentiation of F with y viewed as an independent variable. Once one guesses this shape of expression, it is not difficult to verify it using induction.

Using (5.13), (5.12), and (5.5), we derive

$$\frac{\partial^{a_2+\dots+a_d}}{\partial t_2^{a_2} \dots \partial t_d^{a_d}} F(t_0, t_2, \dots, t_d) \ll \max \frac{X^{j_2+\dots+j_d+k_2+\dots+k_d}}{Z^{j_2+\dots+j_d} X_2^{k_2} \dots X_d^{k_d}} \frac{Z^N}{\prod_{\ell=2}^d X_\ell^{(\mathbf{b}_1+\dots+\mathbf{b}_N) \cdot \mathbf{e}_\ell}}.$$

Since $N \leq j_2 + \dots + j_d$, in the Z -aspect, the above bound is $\ll 1$. The power on X is at most $a_2 + \dots + a_d$, and the power of X_ℓ in the denominator is a_ℓ . Hence

$$\frac{\partial^{a_2+\dots+a_d}}{\partial t_2^{a_2} \dots \partial t_d^{a_d}} F(t_0, t_2, \dots, t_d) \ll \left(\frac{X}{X_2}\right)^{a_2} \dots \left(\frac{X}{X_d}\right)^{a_d},$$

which is precisely the desired condition to show that F is X -inert. \square

Proof of Lemma 5.6. Let $f = \phi'$ (the derivative with respect to the first variable, t_1), so t_0 is defined implicitly by $f(t_0, t_2, \dots, t_d) = 0$. Note that (5.4) translates to

$$f^{(a_1, a_2, \dots, a_d)}(t_1, t_2, \dots, t_d) \ll \frac{Y}{Z} \frac{1}{Z^{a_1}} \left(\frac{X}{X_2}\right)^{a_2} \dots \left(\frac{X}{X_d}\right)^{a_d}.$$

Likewise, the condition $\phi''(t) \gg Y/Z^2$ means

$$f^{(1,0,\dots,0)}(t_1, t_2, \dots, t_d) \gg \frac{Y}{Z^2}.$$

Implicit differentiation gives

$$t_0^{(\mathbf{e}_{j_0})} = -\frac{f^{(\mathbf{e}_{j_0})}}{f^{(\mathbf{e}_1)}},$$

where $j_0 \in \{2, 3, \dots, d\}$, and \mathbf{e}_j denotes the j -th standard basis vector. From this, we easily deduce $t_0^{(\mathbf{e}_{j_0})} \ll Z \frac{X}{X_{j_0}}$, consistent with $\frac{1}{Z} t_0$ being X -inert. Now we proceed inductively to treat arbitrary derivatives. Let $\mathbf{a} = (a_2, \dots, a_d)$. We have

$$t_0^{(\mathbf{a}+\mathbf{e}_{j_0})} = -\frac{\partial^{a_2+\dots+a_d}}{\partial t_2^{a_2} \dots \partial t_d^{a_d}} \frac{f^{(\mathbf{e}_{j_0})}(t_0, t_2, \dots, t_d)}{f^{(\mathbf{e}_1)}(t_0, t_2, \dots, t_d)}.$$

As shorthand, let $g = f^{(\mathbf{e}_{j_0})}$, and $h = f^{(\mathbf{e}_1)}$. By (5.13), we have

$$\begin{aligned} & \frac{\partial^{a_2+\dots+a_d}}{\partial t_2^{a_2} \dots \partial t_d^{a_d}} \left(\frac{g}{h}\right) \\ &= \sum_{\mathbf{j}+\mathbf{k} \leq \mathbf{a}} \sum_{N \leq j_2+\dots+j_d} \sum_{\mathbf{b}_1+\dots+\mathbf{b}_N+\mathbf{k}=\mathbf{a}} c_{\mathbf{j}_2, \dots, \mathbf{j}_d, \mathbf{k}, N, \mathbf{b}_1, \dots, \mathbf{b}_N} \left(\frac{g}{h}\right)^{(j_2+\dots+j_d, k_2, \dots, k_d)} t_0^{(\mathbf{b}_1)} \dots t_0^{(\mathbf{b}_N)}. \end{aligned}$$

Note the total “degree” of any \mathbf{b}_i is at most that of \mathbf{a} , so our inductive hypothesis gives the desired bound for these $t_0^{(\mathbf{b}_i)}$.

We claim that

$$(5.14) \quad \left(\frac{g}{h}\right)^{(\alpha, k_2, \dots, k_d)} \ll Z \frac{X}{X_{j_0}} \frac{1}{Z^\alpha} \left(\frac{X}{X_2}\right)^{k_2} \dots \left(\frac{X}{X_d}\right)^{k_d}.$$

Taking this for granted for a moment, we derive

$$t_0^{(\mathbf{a}+\mathbf{e}_{j_0})} \ll \sum_{\mathbf{j}+\mathbf{k} \leq \mathbf{a}} \sum_{N \leq j_2+\dots+j_d} \sum_{\mathbf{b}_1+\dots+\mathbf{b}_N+\mathbf{k}=\mathbf{a}} Z \frac{X}{X_{j_0}} \frac{1}{Z^{j_2+\dots+j_d}} \left(\frac{X}{X_2}\right)^{k_2} \dots \left(\frac{X}{X_d}\right)^{k_d} Z^N \prod_{\ell=2}^d \left(\frac{X}{X_\ell}\right)^{(\mathbf{b}_1+\dots+\mathbf{b}_N) \cdot \mathbf{e}_\ell},$$

using the inductive hypothesis. This simplifies as

$$t_0^{(\mathbf{a}+\mathbf{e}_{j_0})} \ll Z \frac{X}{X_{j_0}} \left(\frac{X}{X_2}\right)^{a_2} \dots \left(\frac{X}{X_d}\right)^{a_d},$$

as desired.

Now we prove the claim (5.14). The generalized product rule gives

$$\left(\frac{g}{h}\right)^{(\mathbf{m})} = \sum_{\mathbf{m}_1+\mathbf{m}_2=\mathbf{m}} c_{\mathbf{m}_1,\mathbf{m}_2} g^{(\mathbf{m}_1)} \left(\frac{1}{h}\right)^{(\mathbf{m}_2)}.$$

Meanwhile, the derivatives of $1/h$ are given by (5.10). Therefore,

$$\begin{aligned} \left(\frac{g}{h}\right)^{(\alpha,k_2,\dots,k_d)} &\ll \sum_{\mathbf{m}_1+\mathbf{m}_2=(\alpha,k_2,\dots,k_d)} \frac{Y}{Z} \frac{X}{X_{j_0}} \frac{1}{Z^{\mathbf{m}_1 \cdot \mathbf{e}_1}} \prod_{\ell=2}^d \left(\frac{X}{X_\ell}\right)^{\mathbf{m}_1 \cdot \mathbf{e}_\ell} \\ &\times \sum_{\mathbf{j}_1+\dots+\mathbf{j}_{\mathbf{m}_2 \cdot \mathbf{1}}=\mathbf{m}_2} \frac{\left(\frac{Y}{Z^2}\right)^{\mathbf{m}_2 \cdot \mathbf{1}}}{\left(\frac{Y}{Z^2}\right)^{\mathbf{m}_2 \cdot \mathbf{1}+1}} \frac{1}{Z^{(\mathbf{j}_1+\dots+\mathbf{j}_{\mathbf{m}_2 \cdot \mathbf{1}}) \cdot \mathbf{e}_1}} \prod_{\ell=2}^d \left(\frac{X}{X_\ell}\right)^{(\mathbf{j}_1+\dots+\mathbf{j}_{\mathbf{m}_2 \cdot \mathbf{1}}) \cdot \mathbf{e}_\ell}. \end{aligned}$$

This simplifies to give the claimed bound. \square

5.5. A convention. We shall often find it convenient to re-normalize a family of inert functions. For a simple example to illustrate what is meant here, say $w_T(x)$ is X -inert, and supported on $x \asymp N$. We can write $x^{-1/2}w_T(x) = N^{-1/2}W_T(x)$, where $W_T(x) = (x/N)^{-1/2}w_T(x)$. Then W_T forms an inert family, and satisfies the same derivative bounds as w_T , but with a different list of constants $C(j)$. When doing this too many times it becomes difficult to find notation for all the new functions that arise, so we may on occasion replace W_T by w_T , which is supposed to represent a generic inert function.

Another useful convention is, when focusing only a particular variable (say n), we may write $w_N(n, \cdot)$ where the \cdot is a placeholder for the remaining variables which are suppressed in the notation. Writing all the variables would be very unwieldy, and the notion of inertness keeps track of the important behavior of the weight function with respect to the remaining variables.

We will also say that a family of inert functions $\{w_T(x_1, \dots, x_d)\}$ such that each variable x_i is supported in $[X_i, 2X_i]$ is *very small* to mean a quantity which is of size $O_A((X_1 \dots X_d q)^{-A})$ for every $A > 0$, and uniformly in the family $T \in \mathcal{F}$. More generally, we will use this terminology “very small” for more general quantities, not just inert functions. In practice, we will largely ignore very small error terms.

6. FIRST POISSON SUMMATION

Now we return to the analysis of \mathcal{S} from (3.7). We open up the divisor functions using the formulas $\sum_m \tau_2(m)f(m) = \sum_{m_1, m_2} f(m_1 m_2)$, and the definition of $\tau_3(n, F_{a, \sqrt{q}})$ (cf. (3.4)).

6.1. Dyadic partition of unity. Throughout this paper we shall apply dyadic decompositions of important variables. Call a number N *dyadic* if $N = 2^{k/2}$, for some integer k . A dyadic partition of unity is a partition of unity of the form $1 = \sum_{k \in \mathbb{Z}} \omega(2^{-k/2}x)$, where ω is a fixed smooth function with support on the dyadic interval $[1, 2]$. The family $\omega_N(x) = \omega(x/N)$ forms a 1-inert family of functions. Applying this to \mathcal{S} , we have

$$(6.1) \quad \mathcal{S} = \sum_{M_1, M_2, N_1, N_2, N_3, C \text{ dyadic}} \mathcal{S}_{M_1, M_2, N_1, N_2, N_3, C},$$

where the dyadic numbers are restricted to be $\geq 2^{-1/2}$, and where

$$(6.2) \quad \mathcal{S}_{M_1, M_2, N_1, N_2, N_3, C} = \sum_{(a, q)=1}^{\infty} \frac{\mu(a)}{a^{3/2}} \sum_{\substack{c \equiv 0 \pmod{q} \\ c \leq C}} \frac{1}{c} \sum_{n_1, n_2, n_3, m_1, m_2} \frac{S(m_1 m_2, n_1 n_2 n_3 a, c)}{\sqrt{m_1 m_2 n_1 n_2 n_3}} \\ \times J_{\kappa-1} \left(\frac{4\pi \sqrt{m_1 m_2 n_1 n_2 n_3 a}}{c} \right) V \left(\frac{m_1 m_2}{q} \right) F_a \left(\frac{n_1}{\sqrt{q}}, \frac{n_2}{\sqrt{q}}, \frac{n_3}{\sqrt{q}} \right) w_T(m_1, m_2, n_1, n_2, n_3, c).$$

The letter T here and throughout stands for the tuple of dyadic parameters, and we may use \mathcal{S}_T as shorthand for the left hand side of (6.2). For the main thrust of the argument, the precise form of w_T is not important. However, when calculating certain potential main terms, we have found it important to re-sum over the partition, in which case one should remember that w_T may be expressed as

$$w_T(m_1, m_2, n_1, n_2, n_3, c) = \omega \left(\frac{m_1}{M_1} \right) \dots \omega \left(\frac{c}{C} \right).$$

Let $M = M_1 M_2$, and $N = N_1 N_2 N_3$.

Lemma 6.1. *Unless*

$$(6.3) \quad M \ll q^{1+\varepsilon} \quad \text{and} \quad N_i \ll \frac{q^{1/2+\varepsilon}}{a},$$

for all $i = 1, 2, 3$, then we have

$$\mathcal{S}_T \ll_A q^{-A},$$

for $A > 0$ arbitrarily large. Moreover, if $C > q^3$, we have

$$\mathcal{S}_T \ll q^\varepsilon.$$

We shall henceforth assume (6.3) (which implies $N \ll a^{-3} q^{3/2+\varepsilon}$), and

$$(6.4) \quad C \leq q^3.$$

Proof. The bounds (6.3) follow from the rapid decay of the weight functions in the approximate functional equations. The bound for $C > q^3$ holds using the Weil bound for Kloosterman sums, and $J_{\kappa-1}(x) \ll x$. \square

By symmetry (Dirichlet's hyperbola method), we shall assume

$$(6.5) \quad M_1 \leq M_2.$$

Note $M_1 \ll q^{1/2+\varepsilon}$.

6.2. First Poisson summation. Applying Poisson summation in m_2 modulo c , we obtain

$$(6.6) \quad \mathcal{S}_T = \sum_{(a,q)=1} \frac{\mu(a)}{a^{3/2}} \sum_{m_1, n_1, n_2, n_3} \frac{1}{\sqrt{m_1 n_1 n_2 n_3}} \sum_{c \equiv 0 \pmod{q}} \frac{1}{c^2} \sum_{k \in \mathbb{Z}} H(k) I(k),$$

where (with shorthand $n = n_1 n_2 n_3$)

$$(6.7) \quad H(k) = H(k, m_1, na; c) = \sum_{x \pmod{c}} S(m_1 x, na; c) e\left(\frac{kx}{c}\right),$$

and

$$(6.8) \quad I(k) = I(m_1, k, n_1, n_2, n_3, a, c) = \int_0^\infty e\left(\frac{-kt}{c}\right) J_{\kappa-1}\left(\frac{4\pi\sqrt{m_1 nat}}{c}\right) w_{M_2}(t, \cdot) \frac{dt}{\sqrt{t}}.$$

For the notation $w_{M_2}(t, \cdot)$, recall the convention described in Section 5.5. Furthermore note that the definition of the inert function w_T has been altered to include the function V and F_a from the second line of (6.2).

We now apply a dyadic partition of unity to the k -sum, and let $\omega(k/K)$ be a generic such piece. To ease the notation, we simply add K to the long tuple of parameters already appearing in (6.2); we are already writing T as shorthand for this long tuple, and we shall continue this practice. Let $I_K = I_{(M_1, M_2, N_1, N_2, N_3, a, C, K)} = \omega(k/K)I$. Then for $k > 0$, $I = \sum_{K \text{ dyadic}} I_K$. By re-defining the inert function in (6.8) to incorporate $\omega(k/K)$, we may also view I_K as an instance of (6.8).

Remark. We may without loss of generality assume that $k > 0$. The negative values of k give rise to terms that are complex conjugates of their positive counterparts. Secondly, $H(0) = 0$ unless $c|m_1$, and those terms only contribute $O(q^{-A})$ since $m_1 \ll q^{1/2+\varepsilon}$ and $q|c$.

6.3. The arithmetic function. We now compute the arithmetic sum $H(k)$. Immediately from the definition, we obtain

$$(6.9) \quad \frac{1}{c} H(k) = \frac{1}{c} \sum_{u \pmod{c}}^* \sum_{x \pmod{c}} e\left(\frac{(m_1 u + k)x + na\bar{u}}{c}\right) = \sum_{u \pmod{c}}^* \delta_{m_1 u \equiv -k \pmod{c}} e\left(\frac{na\bar{u}}{c}\right).$$

One would like to simply substitute $u \equiv -k\bar{m}_1 \pmod{c}$, however this is not possible because it is not guaranteed that m_1 (or k) are coprime to c . For this reason, we employ a factorization of c and the Chinese remainder theorem as follows.

Write

$$(6.10) \quad c = c_0 c_2, \quad \text{and} \quad k = k_0 k_1,$$

where the factorizations may be written locally, using the notation $\nu_p(n) = d$ for $p^d || n$, as

$$\begin{aligned} c_0 &= \prod_{\nu_p(c) > \nu_p(k)} p^{\nu_p(c)}, & c_2 &= \prod_{1 \leq \nu_p(c) \leq \nu_p(k)} p^{\nu_p(c)}, \\ k_0 &= \prod_{\nu_p(k) \geq \nu_p(c)} p^{\nu_p(k)}, & k_1 &= \prod_{1 \leq \nu_p(k) < \nu_p(c)} p^{\nu_p(k)}. \end{aligned}$$

Alternatively, using the notation $n^* = \prod_{p|n} p$ we have

$$(6.11) \quad (c_0, k_0) = 1, \quad c_2 | k_0, \quad k_1 k_1^* | c_0,$$

and these conditions characterize c_0, c_2, k_0, k_1 . Note that $(c_2, k_1) = 1$ automatically from the other conditions, indeed we also have $(c_2, c_0) = (k_1, k_0) = 1$.

The congruence condition $m_1 u \equiv -k \pmod{c}$ in (6.9) is solvable with $(u, c) = 1$ if and only if $(m_1, c) = (k, c)$. The conditions (6.10), (6.11) give $(k, c) = k_1 c_2$, and so we impose the condition $k_1 c_2 = (m_1, c_0 c_2) = (m_1, \frac{c_0}{k_1} k_1 c_2)$. Thus we define

$$(6.12) \quad m_1 = k_1 c_2 m'_1,$$

where the new variable m'_1 is only subject to the restriction

$$\left(m'_1, \frac{c_0}{k_1}\right) = 1 \Leftrightarrow (m'_1, c_0) = 1,$$

where we have used that c_0/k_1 shares the same prime factors as c_0 .

Remark. In \mathcal{S} , we have $q|c$. If $q|k_1 c_2$, this means $q|m_1$, but we have $m_1 \ll q^{1/2+\varepsilon}$, so the condition $q|c$ may be freely replaced with $q|c_0$, and we may assume

$$(6.13) \quad (q, k_1) = 1.$$

Proposition 6.2. *Given the notation above,*

$$(6.14) \quad \frac{1}{c} H(k, m_1, an; c) = e\left(-\frac{nam'_1 \overline{k_0}}{c_0}\right) S(na, 0; c_2) k_1 \delta_{k_1|na} \delta_{(m'_1, c_0)=1} \delta_{c,k},$$

where $\overline{k_0}$ indicates multiplicative inverse modulo c_0 and where $\delta_{c,k} = 1$ if (6.10) and (6.11) hold, and $\delta_{c,k} = 0$ otherwise.

Proof. First, we note that $m_1 u \equiv -k \pmod{c}$ (that is, $m'_1 k_1 c_2 u \equiv -k_0 k_1 \pmod{c_0 c_2}$) is equivalent to $m'_1 u \equiv -\frac{k_0}{c_2} \pmod{c_0/k_1}$. In other words,

$$(6.15) \quad \overline{u} \equiv -m'_1 \overline{(k_0/c_2)} \pmod{c_0/k_1}.$$

Here $\overline{(k_0/c_2)}$ can be taken to be the multiplicative inverse modulo c_0 , since every prime that divides c_0 also divides c_0/k_1 (via (6.11))

Now we apply the Chinese remainder theorem to the pair c_0 and c_2 , giving

$$\begin{aligned} \frac{1}{c} H(k, m_1, an; c) &= \sum_{u \pmod{c}}^* \delta_{\overline{u} \equiv -m'_1 \overline{(k_0/c_2)} \pmod{\frac{c_0}{k_1}}} e\left(\frac{na \overline{u} (c_0 \overline{c_0} + c_2 \overline{c_2})}{c_0 c_2}\right) \\ &= \sum_{u \pmod{c_2}}^* e\left(\frac{na \overline{u} \overline{c_0}}{c_2}\right) \sum_{u \pmod{c_0}}^* \delta_{\overline{u} \equiv -m'_1 \overline{(k_0/c_2)} \pmod{\frac{c_0}{k_1}}} e\left(\frac{na \overline{u} \overline{c_2}}{c_0}\right). \end{aligned}$$

The sum modulo c_2 is a Ramanujan sum, and for the sum modulo c_0 we replace u by \overline{u} , giving

$$\frac{1}{c} H(k, m_1, an; c) = S(na, 0; c_2) \sum_{u \pmod{c_0}}^* \delta_{\overline{u} \equiv -m'_1 \overline{(k_0/c_2)} \pmod{\frac{c_0}{k_1}}} e\left(\frac{na u \overline{c_2}}{c_0}\right).$$

The congruence restriction on u modulo c_0 may be expressed as

$$(6.16) \quad u \equiv -m'_1 \overline{(k_0/c_2)} + v \frac{c_0}{k_1} \pmod{c_0}, \quad \text{with} \quad v \pmod{k_1}.$$

Here v runs over *all* residue classes modulo k_1 , because as long as u is coprime to c_0/k_1 it is also coprime to c_0 . Thus

$$\begin{aligned} \frac{1}{c} H(k, m_1, an; c) &= S(na, 0; c_2) \sum_{v \pmod{k_1}} e\left(\frac{na(-m'_1(\overline{k_0/c_2}) + v\frac{c_0}{k_1})\overline{c_2}}{c_0}\right) \\ &= S(na, 0; c_2) e\left(-\frac{nam'_1\overline{k_0}}{c_0}\right) k_1 \delta_{k_1|na}. \end{aligned} \quad \square$$

Inserting the conclusion of Proposition 6.2 into (6.6), and imposing (6.13), we get

$$\begin{aligned} (6.17) \quad \mathcal{S}_T &= \sum_{(a,q)=1} \frac{\mu(a)}{a^{3/2}} \sum_{m'_1, n_1, n_2, n_3} \frac{1}{\sqrt{m'_1 n_1 n_2 n_3}} \sum_{\substack{(c_0, m'_1)=1 \\ c_0 \equiv 0 \pmod{q}}} \sum_{(k_0, c_0)=1} \sum_{\substack{k_1 | \frac{c_0}{k_1^*} \\ (k_1, q)=1}} k_1^{1/2} \delta_{k_1|na} \\ &\quad \sum_{c_2|k_0} \frac{1}{c_0 c_2^{3/2}} e\left(-\frac{n_1 n_2 n_3 a m'_1 \overline{k_0}}{c_0}\right) S(n_1 n_2 n_3 a, 0; c_2) I(m'_1 k_1 c_2, k_0 k_1, n_1, n_2, n_3, a, c_0 c_2), \end{aligned}$$

plus a small error term.

6.4. Analysis of integral transform. The asymptotic behavior of I_K depends on if $\frac{\sqrt{aMN}}{C} \gg q^\varepsilon$ or not, since this dictates whether the Bessel function is oscillatory or not.

Lemma 6.3 (Pre-Transition). *Let $I_K(k)$ be defined via (6.8). If*

$$(6.18) \quad \frac{\sqrt{aMN}}{C} \ll q^\varepsilon,$$

then

$$(6.19) \quad M_2^{1/2} I_K(k) = \left(\frac{\sqrt{aMN}}{C}\right)^{\kappa-1} M_2 w_T(\cdot),$$

where $w_T(\cdot)$ is an X -inert function with $X \ll q^{\varepsilon'}$. Furthermore, I_K is very small unless

$$(6.20) \quad \frac{KM_2}{C} \ll q^\varepsilon.$$

Lemma 6.4 (Post-Transition). *If*

$$(6.21) \quad \frac{\sqrt{aMN}}{C} \gg q^\varepsilon,$$

then

$$(6.22) \quad M_2^{1/2} I_K(k) = \frac{CM_2}{(aMN)^{1/2}} e\left(\frac{m_1 na}{ck}\right) w_T(\cdot) + O((kq)^{-A}),$$

where now w_T is a $q^{\varepsilon'}$ -inert function that is very small unless

$$(6.23) \quad K \asymp \frac{(aMN)^{1/2}}{M_2}.$$

Proof of Lemma 6.3. Suppose that (6.18) holds. Then the Bessel function is not oscillatory, and $J_{\kappa-1}(x) = x^{\kappa-1}W(x)$ where W satisfies the same derivative bounds as an X -inert function, with $X \ll q^\varepsilon$ (note that if $1 \ll x \ll q^\varepsilon$, it is still valid to factor out $x^{\kappa-1}$ though there is a small loss of efficiency by doing so). Then by the discussion in Sections 5.2 and 5.5, we have

$$(6.24) \quad M_2^{1/2} I_K(k) = \left(\frac{\sqrt{aMN}}{C} \right)^{\kappa-1} M_2 w_T(\cdot),$$

and that $I_K(k) \ll (kq)^{-A}$ if $K \gg \frac{C}{M_2} q^\varepsilon$. Here w_T is X -inert with $X \ll q^{\varepsilon'}$. \square

Proof of Lemma 6.4. Now suppose that (6.21) holds. Then we use that for $x \gg 1$, we have

$$J_{\kappa-1}(x) = \sum_{\pm} x^{-1/2} e^{\pm ix} W_{\pm}(x),$$

where W_{\pm} satisfies the same derivative bounds as a 1-inert function. Thus

$$\sqrt{M_2} I_K(k) = \sum_{\pm} \frac{C^{1/2}}{(aMN)^{1/4}} \int_{-\infty}^{\infty} w_{M_2}(t, \cdot) e\left(\frac{-kt}{c}\right) e\left(\frac{\pm 2\sqrt{tm_1 na}}{c}\right) dt,$$

where $w_T(t)$ is q^ε -inert (in all previously-declared variables), and supported on $t \asymp M_2$.

Since $k > 0$, if the \pm sign is $-$, then Lemma 5.4 part 1 shows that the integral is very small. Therefore, we focus on the case where the \pm sign is $+$, in which case we obtain an oscillatory integral with phase

$$\phi(t) = -\frac{kt}{c} + \frac{2\sqrt{tm_1 na}}{c}.$$

We have

$$\phi'(t) = -\frac{k}{c} + \frac{\sqrt{m_1 na}}{c\sqrt{t}}, \quad \phi''(t) = -\frac{\sqrt{m_1 na}}{2ct^{3/2}}.$$

There is a unique point t_0 where $\phi'(t_0) = 0$, namely

$$t_0 = \frac{m_1 na}{k^2}.$$

If it is not the case that $t_0 \asymp M_2$ (with large but absolute implied constants), then we have $|\phi'(t)| \gg \frac{\sqrt{aMN}}{cM_1}$ throughout the support of the weight function, and Lemma 5.4 part 1 again shows the integral is small. If $t_0 \asymp M_2$, then the location of t_0 is compatible with the support of ϕ , and stationary phase (Lemma 5.4) shows that

$$\int_{-\infty}^{\infty} w_T(t) e\left(\frac{-kt}{c}\right) e\left(\frac{2\sqrt{tm_1 na}}{c}\right) dt = \frac{C^{1/2} M_2}{(aMN)^{1/4}} e\left(\frac{m_1 na}{ck}\right) w_T\left(\frac{m_1 na}{k^2}, \cdot\right) + O((kq)^{-A}),$$

where w_T on the right hand side is q^ε -inert, and supported on $m_1 na/k^2 \asymp M_2$. \square

7. RECIPROCITY AND OTHER ARITHMETICAL MANIPULATIONS

Next we reorder the summation \mathcal{S}_T in (6.17). We bring the sum over $n = n_1 n_2 n_3$ to the inside, and open up the Ramanujan sum $S(na, 0; c_2) = \sum_{d|(na, c_2)} d\mu(c_2/d)$. This gives

$$(7.1) \quad \mathcal{S}_T = \sum_{(a, q)=1} \frac{\mu(a)}{a^{3/2}} \sum_{c_2} \frac{1}{c_2^{3/2}} \sum_{d|c_2} d\mu(c_2/d) \sum_{(k_1, q)=1} k_1^{1/2} \sum_{m'_1} \frac{1}{\sqrt{m'_1}} \mathcal{S}' + O(q^{-A}),$$

where

$$\begin{aligned} \mathcal{S}' = & \sum_{\substack{(c_0, m'_1)=1 \\ c_0 \equiv 0 \pmod{qk_1k_1^*}}} \frac{1}{c_0} \sum_{\substack{(k_0, c_0)=1 \\ k_0 \equiv 0 \pmod{c_2}}} \\ & \times \sum_{\substack{n_1 n_2 n_3 a \equiv 0 \pmod{k_1} \\ n_1 n_2 n_3 a \equiv 0 \pmod{d}}} \frac{e\left(-\frac{n_1 n_2 n_3 a m'_1 \overline{k_0}}{c_0}\right)}{\sqrt{n_1 n_2 n_3}} I_K(m'_1 k_1 c_2, k_0 k_1, n_1, n_2, n_3, a, c_0 c_2). \end{aligned}$$

We shall not obtain any significant cancellation in the outer summation variables appearing in \mathcal{S} (except for a “fake” main term, in Section 13.7), but substantial cancellation is required in c_0, k_0 , and the n_i .

Note that since $d|c_2$, $c_2|k_0$, $k_1|c_0$, and $(c_0, k_0) = 1$, we have that $(d, k_1) = 1$. Then the congruences in the sum over $n = n_1 n_2 n_3$ are equivalent to $an \equiv 0 \pmod{dk_1}$, which in turn is equivalent to $n \equiv 0 \pmod{\delta_1}$, where

$$(7.2) \quad \delta_1 = \frac{k_1 d}{(a, k_1 d)}.$$

Note that $(\delta_1, q) = 1$, and $(k_0, q) = 1$.

Since $(c_0, k_0) = 1$, we have the reciprocity formula

$$-\frac{\overline{k_0}}{c_0} \equiv \frac{\overline{c_0}}{k_0} - \frac{1}{c_0 k_0} \pmod{1}.$$

Define

$$\begin{aligned} (7.3) \quad J(n_1, n_2, n_3, a, m'_1, c_0, k_0, c_2, k_1) &= e\left(-\frac{nam'_1}{c_0 k_0}\right) I_K(m'_1 k_1 c_2, k_0 k_1, n_1, n_2, n_3, a, c_0 c_2) \\ &= e\left(-\frac{nam_1}{ck}\right) I_K(m_1, k, n_1, n_2, n_3, a, c). \end{aligned}$$

Our next goal is to apply Poisson summation in the n -variables, and to do that we need some preparatory moves. First, consider a formal sum of the form

$$(7.4) \quad \sum_{\substack{m_1, m_2, m_3 \geq 1 \\ m_1 m_2 m_3 \equiv 0 \pmod{r}}} J(m_1, m_2, m_3).$$

The product $m_1 m_2 m_3$ runs over integers of the form mr with $m \geq 1$. Now define

$$r_1 = (m_1, r), \quad m_1 = m'_1 r_1,$$

so $(m'_1, \frac{r}{r_1}) = 1$. Then we have $m'_1 m_2 m_3 = \frac{r}{r_1} m$. Continuing this process, define $r_2 = (m_2, \frac{r}{r_1})$, $m_2 = m'_2 r_2$, so we have $m'_1 m'_2 m_3 = \frac{r}{r_1 r_2} m$ with $(m'_2, \frac{r}{r_1 r_2}) = 1$. Finally, let $r_3 = (m_3, \frac{r}{r_1 r_2})$, and set $m_3 = m'_3 r_3$, whence $(m'_3, \frac{r}{r_1 r_2 r_3}) = 1$. Now we have $m'_1 m'_2 m'_3 = \frac{r}{r_1 r_2 r_3} m$, and the coprimality conditions mean that $(m'_1 m'_2 m'_3, \frac{r}{r_1 r_2 r_3}) = 1$, so $r_1 r_2 r_3 = r$.

Therefore, translating this discussion into formulas, we have that (7.4) equals

$$\sum_{r_1 r_2 r_3 = r} \sum_{(m'_1, r_2 r_3) = 1} \sum_{(m'_2, r_3) = 1} \sum_{m'_3} J(r_1 m'_1, r_2 m'_2, r_3 m'_3).$$

Using Möbius inversion, we have that (7.4) equals

$$(7.5) \quad \sum_{r_1 r_2 r_3 = r} \sum_{e_1 | r_2 r_3} \mu(e_1) \sum_{e_2 | r_3} \mu(e_2) \sum_{n_1, n_2, n_3 \geq 1} J(r_1 e_1 n_1, r_2 e_2 n_2, r_3 n_3).$$

Applying this formula to \mathcal{S}' , we obtain

$$(7.6) \quad \mathcal{S}' = \sum_{r_1 r_2 r_3 = \delta_1} \sum_{e_1 | r_2 r_3} \sum_{e_2 | r_3} \mu(e_1) \mu(e_2) \mathcal{S}'',$$

where

$$\begin{aligned} \mathcal{S}'' = & \sum_{\substack{(c_0, m'_1)=1 \\ c_0 \equiv 0 \pmod{qk_1 k_1^*}}} \frac{1}{c_0} \sum_{\substack{(k_0, c_0)=1 \\ k_0 \equiv 0 \pmod{c_2}}} \sum_{n_1, n_2, n_3 \geq 1} \frac{e\left(\frac{e_1 e_2 \delta_1 a m'_1 n_1 n_2 n_3 \overline{c_0}}{k_0}\right)}{\sqrt{\delta_1 e_1 e_2 n_1 n_2 n_3}} \\ & \times J(r_1 e_1 n_1, r_2 e_2 n_2, r_3 n_3, a, m'_1, c_0, k_0, c_2, k_1). \end{aligned}$$

We remark that in doing so we changed variables by

$$(7.7) \quad n_1 \rightarrow r_1 e_1 n_1, \quad n_2 \rightarrow r_2 e_2 n_2, \quad n_3 \rightarrow r_3 n_3.$$

With the earlier definition of n as $n_1 n_2 n_3$, then (7.7) is equivalent to $n \rightarrow e_1 e_2 \delta_1 n$.

Next define $g_0 = (e_1 e_2 \delta_1 a m'_1, k_0)$, and write

$$k_0 = g_0 k'_0, \quad \text{and} \quad \delta_2 = \frac{e_1 e_2 \delta_1 a m'_1}{g_0}.$$

There are some implicit conditions on the variables that we wish to record explicitly. Note that since $(k_1, k_0) = 1$, and $d | c_2 | k_0$, we may write g_0 as

$$g_0 = (e_1 e_2 \frac{ad}{(a, d)} m'_1, k_0) = d(e_1 e_2 \frac{a}{(a, d)} m'_1, \frac{k_0}{d}),$$

and in particular $d | g_0$, a property that will be important in Section 11.7. Also note that since none of the factors of δ_2 are divisible by q (since q is prime, $(a, q) = 1$, and the original m_1 and n_i -variables are $\ll q^{1/2+\varepsilon}$), we have

$$(7.8) \quad (\delta_2, q) = 1.$$

We may also observe that

$$(g_0, qk_1) = 1,$$

since $g_0 | k_0$, $(k_0, c_0) = 1$, and $c_0 \equiv 0 \pmod{qk_1 k_1^*}$. From $k_1 | a \delta_1$, we also conclude that

$$(7.9) \quad k_1 | \delta_2.$$

Therefore,

$$(7.10) \quad \mathcal{S}'' = \sum_{\substack{g_0 | e_1 e_2 \delta_1 a m'_1 \\ g_0 \equiv 0 \pmod{d}}} \mathcal{S}''',$$

where

$$(7.11) \quad \mathcal{S}''' = \sum_{\substack{(c_0, g_0 m'_1)=1 \\ c_0 \equiv 0 \pmod{q k_1 k_1^*}}} \frac{1}{c_0} \sum_{\substack{(k'_0, \delta_2 c_0)=1 \\ k'_0 \equiv 0 \pmod{\frac{c_2}{(g_0, c_2)}}}} \sum_{n_1, n_2, n_3 \geq 1} \frac{e\left(\frac{\delta_2 n_1 n_2 n_3 \overline{c_0}}{k'_0}\right)}{\sqrt{\delta_1 e_1 e_2 n_1 n_2 n_3}} \\ \times J(r_1 e_1 n_2, r_2 e_2 n_2, r_3 n_3, a, m'_1, c_0, g_0 k'_0, c_2, k_1).$$

8. TRIPLE POISSON

8.1. Statement of result.

Proposition 8.1. *Let J be any smooth and compactly-supported function on $(0, \infty)^3$, and suppose $(\alpha, k) = 1$. Then*

$$\sum_{n_1, n_2, n_3 \geq 1} e\left(\frac{n_1 n_2 n_3 \alpha}{k}\right) J(n_1, n_2, n_3) = \frac{1}{k^3} \sum_{p_1, p_2, p_3 \in \mathbb{Z}} A(p_1, p_2, p_3; \alpha; k) B(p_1, p_2, p_3; k),$$

where

$$(8.1) \quad A(p_1, p_2, p_3; \alpha; k) = \sum_{x_1, x_2, x_3 \pmod{k}} e\left(\frac{x_1 x_2 x_3 \alpha - x_1 p_1 - x_2 p_2 - x_3 p_3}{k}\right),$$

and

$$B(p_1, p_2, p_3; k) = \int_0^\infty \int_0^\infty \int_0^\infty J(t_1, t_2, t_3) e\left(\frac{p_1 t_1}{k} + \frac{p_2 t_2}{k} + \frac{p_3 t_3}{k}\right) dt_1 dt_2 dt_3.$$

An evaluation for A is given in Lemma 8.2 (see also Lemma 13.2 for the case when some $p_i = 0$).

In view of (7.11), we require $A(p_1, p_2, p_3; \delta_2 \overline{c_0}; k'_0)$, and

$$(8.2) \quad B(p_1, p_2, p_3; k'_0) = \int_{(\mathbb{R}^+)^3} J(r_1 e_1 t_1, r_2 e_2 t_2, r_3 t_3, a, m'_1, c_0, g_0 k'_0, c_2, k_1) \\ e\left(\frac{p_1 t_1}{k'_0} + \frac{p_2 t_2}{k'_0} + \frac{p_3 t_3}{k'_0}\right) \frac{dt_1 dt_2 dt_3}{\sqrt{e_1 e_2 \delta_1 t_1 t_2 t_3}},$$

where $r_1 r_2 r_3 = \delta_1$ (see (7.6)).

8.2. The evaluation of A .

Lemma 8.2. *Suppose $(\alpha, k) = 1$. We have*

$$A(p_1, p_2, p_3; \alpha; k) = k \sum_{f|(p_2, p_3, k)} f S\left(p_1 \overline{\alpha}, \frac{p_2 p_3}{f^2}; \frac{k}{f}\right).$$

Proof. By first evaluating the sum over x_3 , we derive

$$A(p_1, p_2, p_3; \alpha; k) = k \sum_{\substack{x_1, x_2 \pmod{k} \\ x_1 x_2 \alpha \equiv p_3 \pmod{k}}} e\left(\frac{x_1 p_1 + x_2 p_2}{k}\right).$$

At this point we decompose the sum by letting $(x_1, k) = f$ with $f|k$. Say $x_1 = fy$ with y running over reduced residue classes modulo k/f . Note that necessarily $f|(p_3, k)$, and that

$x_2 \equiv \overline{\alpha y} \frac{p_3}{f} \pmod{k/f}$. Therefore, we may write $x_2 = \overline{\alpha y} \frac{p_3}{f} + v \frac{k}{f}$ where v runs modulo f . Hence,

$$A(p_1, p_2, p_3; \alpha; k) = k \sum_{f|(p_3, k)} \sum_{y \pmod{k/f}}^* e\left(\frac{yp_1}{k/f}\right) e\left(\frac{\overline{\alpha y} \frac{p_3}{f} p_2}{k}\right) \sum_{v \pmod{f}} e\left(\frac{p_2 v}{f}\right).$$

The sum over v detects $f|p_2$, and so the formula follows. \square

8.3. Asymptotics of B . Let us begin by unraveling the definition of B . First we recall its definition from (8.2), (7.3), and (6.8). Let us also pull out a factor $N^{-1/2}$ coming from $(e_1 e_2 \delta_1 t_1 t_2 t_3)^{-1/2}$. Recall that $I_K(k)$ has a built-in inert function. One may change this inert function appropriately to obtain that B takes a simplified form

$$N^{1/2} B(p_1, p_2, p_3; k'_0) = \int_{(\mathbb{R}^+)^3} e\left(\frac{-e_1 e_2 \delta_1 t_1 t_2 t_3 a m'_1}{c_0 k_0}\right) I^*(m_1, k, r_1 e_1 t_1, r_2 e_2 t_2, r_3 t_3 a, c) \\ e\left(\frac{p_1 t_1}{k'_0} + \frac{p_2 t_2}{k'_0} + \frac{p_3 t_3}{k'_0}\right) dt_1 dt_2 dt_3,$$

where I^* has the same properties as given in Lemmas 6.3 and 6.4 (since all that changed is the definition of the inert function). Note that the support of the inert function is such that $t_i \asymp N'_i$, say, where

$$N'_1 = \frac{N_1}{e_1 r_1}, \quad N'_2 = \frac{N_2}{e_2 r_2}, \quad N'_3 = \frac{N_3}{r_3}.$$

Define $N' = N'_1 N'_2 N'_3$. In the analytic aspects, it is usually most convenient to work with the original variable names (we may perform the substitutions later, after analyzing the integral transform). Let

$$h = e_1 e_2 r_1 r_2 r_3 = e_1 e_2 \delta_1,$$

and note that

$$N' h = N.$$

Next let B_P be the same as B but multiplied by a part of a partition of unity with $\pm p_i \asymp P_i$, $i = 1, 2, 3$.

Lemma 8.3 (Post-Transition). *Suppose (6.21) holds. Then*

$$(8.3) \quad M_2^{1/2} N^{1/2} B_P(p_1, p_2, p_3; k'_0) = \frac{C}{(aMN)^{1/2}} M_2 N' w_T(\cdot),$$

where w_T is q^ε -inert, $N' = N'_1 N'_2 N'_3$, and where $B_P(p_1, p_2, p_3)$ is very small unless

$$(8.4) \quad P_i \ll \frac{k'_0}{N'_i} q^\varepsilon \quad \text{and} \quad K \asymp \frac{(aMN)^{\frac{1}{2}}}{M_2}.$$

Proof of Lemma 8.3. The main observation is that the exponential factor appearing in (6.22) cancels the exponential factor in the definition of J in (7.3). Therefore, B is a Fourier transform of a q^ε -inert function supported on $t_i \asymp N'_i$, and hence by the discussion in Section 5.2, (8.3) follows. \square

Lemma 8.4 (Pre-Transition, non-oscillatory). *Suppose (6.18) holds. If in addition*

$$(8.5) \quad \frac{NaM_1}{CK} \ll q^\varepsilon,$$

then

$$(8.6) \quad M_2^{1/2} N^{1/2} B_P(p_1, p_2, p_3; k'_0) = \left(\frac{\sqrt{aMN}}{C} \right)^{\kappa-1} M_2 N' w_T(\cdot),$$

where the inert function is X -inert with $X \ll q^\varepsilon$, and where w_T is very small unless

$$\frac{N'_i P_i}{k'_0} \ll q^\varepsilon, \quad i = 1, 2, 3, \quad \text{and} \quad \frac{KM_2}{C} \ll q^\varepsilon.$$

Proof. In this case, the exponential factor in the definition of B_P is essentially not oscillatory, because of the condition (8.5). For this, it is again helpful to remember that

$$\frac{e_1 e_2 \delta_1 t_1 t_2 t_3 a m'_1}{c_0 k_0} \asymp \frac{NaM_1}{CK}.$$

Since (8.5) holds, we may include this exponential factor into the definition of the inert weight function; the inert function is X -inert with $X \ll q^\varepsilon$. As in the case of Lemma 8.3, we again obtain a Fourier transform of an X -inert function, and hence we obtain the claimed estimates. \square

We record that under the conditions of Lemmas 8.3 and 8.4, we have

$$(8.7) \quad P_1 P_2 P_3 \ll q^\varepsilon \frac{h}{N} \left(\frac{k}{k_1 g_0} \right)^3 \asymp q^\varepsilon \frac{K^3}{N} \frac{h}{(k_1 g_0)^3}.$$

Lemma 8.5 (Pre-Transition, oscillatory). *Suppose (6.18) holds. If in addition*

$$(8.8) \quad \frac{NaM_1}{CK} \gg q^\varepsilon,$$

then

$$(8.9) \quad M_2^{1/2} N^{1/2} B_P(p_1, p_2, p_3; k'_0) = O(q^{-A} \prod_{i=1}^3 (1 + |p_i|)^{-A}) \\ + \left(\frac{\sqrt{aMN}}{C} \right)^{\kappa-1} M_2 N' \left(\frac{CK}{aM_1 N} \right)^{3/2} e \left(\frac{2(p_1 p_2 p_3 c k)^{1/2}}{(a m_1 h k_0'^3)^{1/2}} \right) w_T(\cdot),$$

where the inert function is X -inert with $X \ll q^\varepsilon$. Moreover, $B_P(p_1, p_2, p_3; k'_0)$ is very small unless each $p_i > 0$ and

$$(8.10) \quad P_i \asymp \frac{NaM_1 k'_0}{CK N'_i}, \quad i = 1, 2, 3 \quad \text{and} \quad \frac{KM_2}{C} \ll q^\varepsilon.$$

Observe the identity

$$\frac{2(p_1 p_2 p_3 c k)^{1/2}}{(a m_1 h k_0'^3)^{1/2}} = \frac{2(p_1 p_2 p_3 c_0)^{1/2}}{k'_0 \left(\frac{a h m'_1}{g_0} \right)^{1/2}}.$$

Proof. In this case, the phase arising from reciprocity is legitimately oscillatory, and is not cancelled by a corresponding phase from the kernel function I_K . By (6.19) and (7.3), we have

$$M_2^{1/2} N^{1/2} B_P = \\ \left(\frac{\sqrt{aMN}}{C} \right)^{\kappa-1} M_2 \int_{\mathbb{R}^3} w_T(t_1, t_2, t_3, \cdot) e \left(\frac{-t_1 t_2 t_3 a m_1 h}{c k} \right) e \left(\frac{t_1 p_1 + t_2 p_2 + t_3 p_3}{k'_0} \right) dt_1 dt_2 dt_3.$$

The oscillation of $e(\frac{-t_1 t_2 t_3 a m_1 h}{ck})$ is such that repeated integration by parts (Lemma 5.4, part (1)) shows $B_P \ll q^{-A} \prod_{i=1}^3 (1 + |p_i|)^{-A}$ unless

$$p_i \asymp \frac{NaM_1 k'_0}{CKN'_i},$$

which gives the final statement of the lemma. In particular, we may assume that $p_i > 0$ for each i .

Next change variables $t_1 = \frac{t}{t_2 t_3}$ (so t is supported on $t \asymp N'$ now), getting

$$M_2^{1/2} N^{1/2} B_P = O(q^{-A} \prod_{i=1}^3 (1 + |p_i|)^{-A}) \\ + \left(\frac{\sqrt{aMN}}{C} \right)^{\kappa-1} \frac{M_2}{N'_2 N'_3} \int_{\mathbb{R}} e\left(\frac{-tam_1 h}{ck}\right) \left(\int_{\mathbb{R}^2} w_T(\cdot) e\left(\frac{\frac{t}{t_2 t_3} p_1 + t_2 p_2 + t_3 p_3}{k'_0}\right) dt_2 dt_3 \right) dt.$$

In the t_2 and t_3 integrals we may use iterated stationary phase (Lemmas 5.4 and 5.5) to develop an asymptotic formula. As a shortcut, the required asymptotic was developed in [36, Lemma 6.4]. We have

$$\int_{\mathbb{R}^2} w_T(\cdot) e\left(\frac{\frac{t}{t_2 t_3} p_1 + t_2 p_2 + t_3 p_3}{k'_0}\right) dt_2 dt_3 = \frac{CKN'_2 N'_3}{NaM_1} e\left(\frac{3(p_1 p_2 p_3 t)^{1/3}}{k'_0}\right) w_T(\cdot),$$

plus a very small error. Therefore,

$$M_2^{1/2} N^{1/2} B_P = \left(\frac{\sqrt{aMN}}{C} \right)^{\kappa-1} M_2 \frac{CK}{NaM_1} \int_{\mathbb{R}} e\left(\frac{-tam_1 h}{ck}\right) e\left(\frac{3(p_1 p_2 p_3 t)^{1/3}}{k'_0}\right) w_T(\cdot) dt,$$

plus a very small error. A final stationary phase analysis in t gives a stationary point at

$$t_0 = (p_1 p_2 p_3)^{1/2} \left(\frac{ck}{k'_0 a m_1 h} \right)^{3/2},$$

which leads immediately to (8.9). \square

8.4. Mellin transform of B . For many of our later purposes, we prefer to work with the Mellin transform of B instead of B itself. Of course, B depends on a number of variables, and what is meant here is the Mellin transform *in terms of* k'_0 . Define

$$(8.11) \quad \tilde{B}(s) := \int_0^\infty B_P(p_1, p_2, p_3; x) x^s \frac{dx}{x},$$

which is the Mellin transform of B_P in k'_0 . Recalling $k = g_0 k_1 k'_0$, note that

$$x \asymp \frac{K}{g_0 k_1}.$$

Let us combine the results from Lemmas 8.3 and 8.4. In these two cases, we have

$$(8.12) \quad M_2^{1/2} N^{1/2} B_P(p_1, p_2, p_3; k'_0) = \left(\frac{\sqrt{aMN}}{C} \right)^\delta M_2 N' w_T(\cdot),$$

where $\delta = -1$ in Lemma 8.3, and $\delta = \kappa - 1$ in Lemma 8.4. In both cases, p_i are supported on $|p_i| \asymp P_i \ll \frac{k'_0}{N'_i} q^\varepsilon$, but there are some different constraints on the parameters. In any event, in terms of k'_0 , it is easy to perform the Mellin transform of B_P in these cases, because B_P is X -inert. We shall group these two cases together under the heading of “Non-oscillatory”.

Lemma 8.6 (Non-Oscillatory). *Suppose the conditions of Lemma 8.3 or Lemma 8.4 hold, and put $\delta = -1$ or $\delta = \kappa - 1$ in the respective cases. Then*

$$M_2^{1/2} N^{1/2} \tilde{B}(s) = \left(\frac{\sqrt{aMN}}{C} \right)^\delta M_2 N' \left(\frac{K}{g_0 k_1} \right)^s w_T(\cdot; s),$$

where w_T is q^ε -inert in all the variables except for s , and entire in terms of s . Moreover, $w_T(\cdot; \sigma + it)$ is very small unless $|t| \ll_\sigma q^\varepsilon$.

In the case that B is oscillatory, it turns out to be easier to use the Bessel integral representation in the Bruggeman-Kuznetsov formula, and so we may happily avoid the Mellin transform analysis of B . See the introductory paragraphs of Section 10.2 for more explanation.

9. APPLICATION OF BRUGGEMAN-KUZNETSOV

Write \mathcal{T}_P for the terms from \mathcal{S}''' with B replaced by B_P (in particular, $p_1 p_2 p_3 \neq 0$). Therefore,

$$\mathcal{T}_P = \sum_{\substack{(c_0, g_0 m'_1)=1 \\ c_0 \equiv 0 \pmod{q k_1 k_1^*}}} \frac{1}{c_0} \sum_{\substack{(k'_0, \delta_2 c_0)=1 \\ k'_0 \equiv 0 \pmod{\frac{c_2}{(g_0, c_2)}}}} \frac{1}{k_0'^3} \sum_{p_1, p_2, p_3 \neq 0} A(p_1, p_2, p_3; \delta_2 \overline{c_0}; k'_0) B_P(p_1, p_2, p_3; k'_0).$$

Applying Lemma 8.2, and moving the sum over k'_0 to the inside, we obtain

$$\mathcal{T}_P = \sum_{\substack{(c_0, g_0 m'_1)=1 \\ c_0 \equiv 0 \pmod{q k_1 k_1^*}}} \frac{1}{c_0} \sum_{p_1, p_2, p_3 \neq 0} \sum_{f | (p_2, p_3)} \sum_{\substack{(k'_0, \delta_2 c_0)=1 \\ k'_0 \equiv 0 \pmod{\frac{c_2}{(g_0, c_2)}} \\ k'_0 \equiv 0 \pmod{f}}} \frac{f}{k_0'^2} S\left(p_1 c_0 \overline{\delta_2}, \frac{p_2 p_3}{f^2}; \frac{k'_0}{f}\right) B_P(p_1, p_2, p_3; k'_0).$$

We can alternatively put the $k_0'^{-2} \asymp \frac{(g_0 k_1)^2}{K^2}$ as part of an inert function which changes the definition of B_P (let us call the new function $B_{P,*}$), but not any of the analytic properties it satisfies (cf. Section 5.5), giving

$$\mathcal{T}_P = \frac{(g_0 k_1)^2}{K^2} \sum_{\substack{(c_0, g_0 m'_1)=1 \\ c_0 \equiv 0 \pmod{q k_1 k_1^*}}} \frac{1}{c_0} \sum_{p_1, p_2, p_3 \neq 0} \sum_{f | (p_2, p_3)} f \sum_{\substack{(k'_0, \delta_2 c_0)=1 \\ k'_0 \equiv 0 \pmod{\frac{c_2}{(g_0, c_2)}} \\ k'_0 \equiv 0 \pmod{f}}} S\left(p_1 c_0 \overline{\delta_2}, \frac{p_2 p_3}{f^2}; \frac{k'_0}{f}\right) B_{P,*}(p_1, p_2, p_3; k'_0).$$

Let $k'_0 = f k''_0$, so that the inner sum over k'_0 becomes

$$\sum_{\substack{\delta_{(f, \delta_2 c_0)}=1 \\ (k''_0, \delta_2 c_0)=1 \\ k''_0 \equiv 0 \pmod{\delta_3}}} S\left(p_1 c_0 \overline{\delta_2}, \frac{p_2 p_3}{f^2}; k''_0\right) B_{P,*}(p_1, p_2, p_3, f k''_0),$$

where we have defined

$$(9.1) \quad \delta_3 = \frac{\frac{c_2}{(g_0, c_2)}}{\left(f, \frac{c_2}{(g_0, c_2)}\right)}.$$

Finally, to ease a later summation over c_0 , we detect the condition $(k_0'', c_0) = 1$ with Möbius inversion, say over the variable δ_4 . Then we reverse the orders of summation, and define

$$(9.2) \quad c_0 = \delta_4 c'_0, \quad \delta_5 = [\delta_3, \delta_4].$$

We record that the summation conditions in the sum over k_0'' are empty unless

$$(\delta_2, \delta_5) = 1 \Leftrightarrow (\delta_2, \delta_3 \delta_4) = 1.$$

For later use, we also record that

$$(9.3) \quad (\delta_4, k_1) = 1,$$

since $k_1 | \delta_2$, and $(\delta_2, \delta_4) = 1$. Using this, and moving the sum over f to the outside, with the definitions

$$p_2 = f p'_2, \quad p_3 = f p'_3,$$

we obtain

$$(9.4) \quad \mathcal{T}_P = \frac{(g_0 k_1)^2}{K^2} \sum_{\delta_3 | \frac{c_2}{(g_0, c_2)}} \sum_{(\delta_4, \delta_2 g_0 m'_1)=1} \frac{\mu(\delta_4)}{\delta_4} \sum_{\substack{(f, \delta_2 \delta_4)=1 \\ (9.1) \text{ is true}}} f \sum_{\substack{(c'_0, f g_0 m'_1)=1 \\ \delta_4 c'_0 \equiv 0 \pmod{q k_1 k_1^*}}} \frac{1}{c'_0} \sum_{p_1, p'_2, p'_3 \neq 0} \mathcal{K},$$

where

$$(9.5) \quad \mathcal{K} = \sum_{\substack{(k_0'', \delta_2)=1 \\ k_0'' \equiv 0 \pmod{\delta_5}}} S(p_1 \delta_4 c'_0 \overline{\delta_2}, p'_2 p'_3; k_0'') B_{P,*}(p_1, f p'_2, f p'_3; f k_0'').$$

Consulting (4.24), we may now realize the Kloosterman sum in question as one belonging to the group $\Gamma = \Gamma_0(\delta_2 \delta_5)$ with the pair of cusps $\infty, \frac{1}{\delta_5}$ (note that these are Atkin-Lehner cusps, since $(\delta_2, \delta_5) = 1$). Hence

$$\mathcal{K} = \sum_{k_0'' \sqrt{\delta_2} \in \mathcal{C}_{\infty, \frac{1}{\delta_5}}} S_{\infty, \frac{1}{\delta_5}}(p_1 \delta_4 c'_0, p'_2 p'_3; k_0'' \sqrt{\delta_2}) B_{P,*}(p_1, f p'_2, f p'_3; f k_0'').$$

According to Theorem 4.7, write $\mathcal{K} = \mathcal{K}_d + \mathcal{K}_c + \mathcal{K}_h$, and accordingly write $\mathcal{T}_P = \mathcal{T}_d + \mathcal{T}_c + \mathcal{T}_h$. We furthermore decompose $\mathcal{K} = \sum_{\epsilon_1, \epsilon_2, \epsilon_3 \in \{-1, 1\}} \mathcal{K}_{\epsilon_1, \epsilon_2, \epsilon_3}$, where the meaning is $\epsilon_i p_i \geq 1$ for $i = 1, 2, 3$, and likewise decompose \mathcal{K}_d , etc. To help ease the notation, let \mathcal{K}^+ denote the terms with $p_i \geq 1$ for all i , and \mathcal{K}^- denote the terms with $p_1 \leq -1$ and $p_2, p_3 \geq 1$. For instance,

$$(9.6) \quad \mathcal{K}_d^\pm = \sum_{t_j \text{ level } \delta_2 \delta_5} \nu_{\infty, j}(p_1 \delta_4 c'_0) \overline{\nu}_{\frac{1}{\delta_5}, j}(p'_2 p'_3) W_\pm(p_1, f p'_2, f p'_3; t_j),$$

where

$$W_\pm(p_1, f p'_2, f p'_3; t_j) = \int_{(2\theta+\epsilon)} h_\pm(s, t_j) \left(4\pi \sqrt{\delta_4 c'_0 |p_1| p'_2 p'_3} \right)^{-s} \widetilde{\widetilde{B}}_{P,*}(p_1, f p'_2, f p'_3; s+1) ds,$$

(recall h_\pm was defined by (4.8)), and where

$$\widetilde{\widetilde{B}}_{P,*}(p_1, f p'_2, f p'_3; s+1) := \int_0^\infty B_{P,*}\left(p_1, f p'_2, f p'_3; \frac{fy}{\sqrt{\delta_2}}\right) y^{s+1} \frac{dy}{y}.$$

Here the “double tilde” notation for B is meant to indicate the Mellin transform of B with respect to $\gamma = k_0'' \sqrt{\delta_2}$ (where $\gamma \in \mathcal{C}_{ab}$ as in (4.30)), because we have already reserved the

meaning of \tilde{B} for the Mellin transform in the k'_0 -variable (as in Section 8.4). The relationship between these two transforms is given by

$$\tilde{B}_{P,*}(s+1) = \left(\frac{\sqrt{\delta_2}}{f}\right)^{s+1} \tilde{B}_{P,*}(s+1).$$

Simplifying, we obtain

$$(9.7) \quad W_{\pm}(p_1, p_2, p_3; t_j) = \frac{\sqrt{\delta_2}}{f} \int_{(2\theta+\varepsilon)} h_{\pm}(s, t_j) \left(\frac{\sqrt{\delta_2}}{\sqrt{\delta_4 c'_0 |p_1| p_2 p_3}}\right)^s \tilde{B}_{P,*}(s+1) ds.$$

The holomorphic case is similar, but with a different integral kernel than $h_{\pm}(s, t_j)$.

We may also prefer to use the Bessel integral representation for W , which we do in case $B_{P,*}$ is oscillatory. For instance, we have

$$W_h(p_1, fp'_2, fp'_3; \ell) = \int_0^{\infty} J_{\ell-1}\left(\frac{4\pi\sqrt{p_1\delta_4 c'_0 p'_2 p'_3}}{y}\right) B_{P,*}\left(p_1, fp'_2, fp'_3; \frac{fy}{\sqrt{\delta_2}}\right) dy.$$

Changing variables, we obtain

$$(9.8) \quad W_h(p_1, fp'_2, fp'_3; \ell) = \frac{\sqrt{\delta_2}}{f} \int_0^{\infty} J_{\ell-1}\left(\frac{4\pi\sqrt{f^2 p_1 p'_2 p'_3 \delta_4 c'_0}}{\sqrt{\delta_2} y}\right) B_{P,*}(p_1, fp'_2, fp'_3; y) dy.$$

Note that, in terms of older variables names, we have

$$\frac{f^2 p_1 p'_2 p'_3 \delta_4 c'_0}{\delta_2} = \frac{p_1 p_2 p_3 c_0}{(ahm'_1)/g_0}.$$

Similarly, in the $+$ Maass case, we have

$$(9.9) \quad W_+(p_1, fp'_2, fp'_3; t_j) = \frac{\sqrt{\delta_2}}{f} \int_0^{\infty} B_{2it_j}^+\left(\frac{4\pi\sqrt{f^2 p_1 p'_2 p'_3 \delta_4 c'_0}}{\sqrt{\delta_2} y}\right) B_{P,*}(p_1, fp'_2, fp'_3; y) dy.$$

10. ASYMPTOTICS OF W

Here we analyze the various W -functions appearing in the Bruggeman-Kuznetsov formula.

10.1. Non-oscillatory cases. First suppose the conditions of Lemma 8.3 or Lemma 8.4 hold, so that Lemma 8.6 gives the behavior of \tilde{B} . Continuing from (9.7), we have

$$(10.1) \quad W_{\pm}(p_1, p_2, p_3; t_j) = \frac{\left(\frac{\sqrt{aMN}}{C}\right)^{\delta} M_2 N' K}{M_2^{1/2} N^{1/2}} \frac{\sqrt{\delta_2}}{f g_0 k_1} \int_{(2\theta+\varepsilon)} h_{\pm}(s, t_j) \left(\frac{\sqrt{\delta_2} K}{g_0 k_1 \sqrt{\delta_4 c'_0 |p_1| p_2 p_3}}\right)^s w_T(s, \cdot) ds.$$

Here w_T is q^{ε} -inert in all variables except s . It is entire in s , with rapid decay for $|\operatorname{Im}(s)| \gg q^{\varepsilon}$.

As shorthand, let

$$(10.2) \quad Y = \frac{g_0 k_1 \sqrt{C P_1 P_2 P_3}}{\sqrt{\delta_2 c_2} K} \asymp \left(\frac{\sqrt{\delta_2} K}{g_0 k_1 \sqrt{\delta_4 c'_0 |p_1| p_2 p_3}}\right)^{-1}.$$

Our goal now is to show

Lemma 10.1 (Non-Oscillatory). *Suppose the conditions of Lemma 8.3 or Lemma 8.4 hold. If $|t_j| \gg (1+Y)q^{\varepsilon}$, then W_{\pm} is very small. Similarly, if $k \gg (1+Y)q^{\varepsilon}$, then W_h is very small.*

Proof. If $s = \sigma + it$, and $|t| \gg (|t_j|q)^\varepsilon$, then by the rapid decay of w_T , we conclude that this part of the integral is bounded in a satisfactory manner. In the complementary region, we then have from Stirling that

$$h_\pm(\sigma + it, t_j) \ll_\sigma q^\varepsilon (q^\varepsilon + |t_j|)^{\sigma-1}.$$

Side remark: The exponential factor implicitly appearing in Stirling's bound on $h_\pm(s, t_j)$ is $\ll 1$, and one cannot do better than this in general, because in one of the two cases of \pm sign, the exponential factor is exactly 1.

Now if $|t_j| \gg (1+Y)q^\varepsilon$, we shift the contour far to the left and bound it trivially. In doing so, one encounters poles at $\frac{s}{2} \pm it_j = 0, -1, -2, \dots$. However, these all have large imaginary part and w_T is very small here, so these residues are bounded in a satisfactory manner. The integral on the new line is very small since $|t_j|/Y \gg q^\varepsilon$.

Next consider $W_h(k)$. The analysis is similar, except one replaces $h_\pm(s, t_j)$ by

$$h(s, k) := 2^{s-1} \frac{\Gamma(\frac{s+k-1}{2})}{\Gamma(\frac{-s+k+1}{2})}.$$

Stirling's formula gives, for $\sigma \ll \sqrt{|k+it|}$, that

$$|h(\sigma + it, k)| \ll (\max(k, |t|))^{\sigma-1}.$$

As before, if $k \gg (1+Y)q^\varepsilon$, then we may move the contour far to the left (some large constant not growing with q). Then we get that W_h is small, by the exact same type of reasoning as in the Maass case. \square

Now we reap the reward of the language of inert functions. Since w_T is inert in all variables, we may apply the Mellin inversion formula, giving

$$(10.3) \quad W_\pm(p_1, p_2, p_3; t_j) = \left(\frac{\sqrt{aMN}}{C} \right)^{\kappa-1} (M_2 N)^{1/2} K \frac{\sqrt{\delta_2}}{h f g_0 k_1} \int_{(2\theta+\varepsilon)} h_\pm(s, t_j) \\ \times \int \left(\frac{\sqrt{\delta_2} K}{g_0 k_1 \sqrt{\delta_4 c'_0 |p_1| p_2 p_3}} \right)^s \widetilde{w}_T(s, \mathbf{u}) \left(\frac{P_1}{|p_1|} \right)^{u_1} \left(\frac{P_2}{p_2} \right)^{u_2} \left(\frac{P_3}{p_3} \right)^{u_3} \left(\frac{C}{\delta_4 c'_0 c_2} \right)^{u_4} d\mathbf{u} ds,$$

plus a small error term. Here \widetilde{w}_T is very small except if the imaginary parts of all the variables are $\ll q^\varepsilon$.

10.2. Oscillatory Case. Now we consider W_\pm and W_h when B is given by Lemma 8.5. The first significant point is that W_- is small, because this corresponds to the case where $p_1 p_2 p_3 < 0$, which means some $p_i < 0$, in which case B is small. Indeed, B is small unless $p_i > 0$ for all i , and so the only relevant functions are W_+ and W_h .

It is inconvenient to use (9.7) in the oscillatory case. The problem is that the oscillatory nature of B means that we may no longer restrict $|\text{Im}(s)|$ to be $O(q^\varepsilon)$, which in turn has an effect on the behavior of $h_+(s, r)$ and $h(s, k)$. Namely, it is no longer true that $h_+(s, r)$ and $h(s, k)$ satisfy analogous asymptotic formulas (due to the use of Stirling with ir large vs. k large), and so it appears difficult to unify these two cases. In addition, one is forced to confront some tricky oscillatory integrals. To sidestep these problems entirely, we shall use the Bessel integral formula for W instead. The oscillatory behavior of B is actually beneficial and causes W to be essentially inert (in both the Maass and holomorphic cases).

Let us begin with W_h . We have

$$W_h(p_1, p_2, p_3; \ell) = \left(\frac{\sqrt{aMN}}{C} \right)^\delta \sqrt{M_2 N} \left(\frac{CK}{aM_1 N} \right)^{3/2} \frac{\sqrt{\delta_2}}{fh} Z,$$

plus a small error term, where Z is shorthand for

$$(10.4) \quad Z = \int_0^\infty J_{\ell-1} \left(\frac{4\pi \sqrt{p_1 p_2 p_3 \delta_4 c'_0}}{\sqrt{\delta_2} y} \right) e \left(\frac{2\sqrt{p_1 p_2 p_3 \delta_4 c'_0}}{\sqrt{\delta_2} y} \right) w_T(y, \cdot) dy.$$

Here we recall that w_T has support on $y \asymp \frac{K}{g_0 k_1}$. The fact that the phases match is pleasant.

Recall the integral representation

$$(10.5) \quad J_{\ell-1}(x) = \sum_{\pm} c_{\ell, \pm} \int_0^{\pi/2} \cos((\ell-1)\theta) e^{\pm i x \cos(\theta)} d\theta, \quad c_{\ell, \pm} = \frac{e^{\mp i(\ell-1)\frac{\pi}{2}}}{\pi}.$$

This gives

$$Z = \sum_{\pm} \int_0^{\pi/2} c_{\ell, \pm} \cos((\ell-1)\theta) \int_0^\infty e \left(\frac{z(1 \pm \cos \theta)}{y} \right) w_T(y, \cdot) dy d\theta,$$

with

$$z = \frac{2\sqrt{p_1 p_2 p_3 \delta_4 c'_0}}{\sqrt{\delta_2}}.$$

Changing variables $y = \frac{K}{g_0 k_1 x}$ now gives $x \asymp 1$, and the inner integral is a Fourier transform of an inert function. Hence

$$Z = \frac{K}{g_0 k_1} \sum_{\pm} \int_0^{\pi/2} c_{\ell, \pm} \cos((\ell-1)\theta) \widehat{w_T} \left(\frac{z g_0 k_1}{K} (1 \pm \cos \theta) \right) d\theta,$$

where we have re-defined w_T (see Section 5.5). Using $\delta_4 c'_0 = c_0 = \frac{c}{c_2}$, $\delta_2 = \frac{ham'_1}{g_0}$, $m'_1 = \frac{m_1}{k_1 c_2}$, (8.10), and $k'_0 = \frac{k}{k_1 g_0}$, we check the size of

$$\frac{z g_0 k_1}{K} \asymp \frac{\sqrt{P_1 P_2 P_3 C k_1^3 g_0^3}}{K \sqrt{ha M_1}} \asymp \frac{Na M_1}{CK},$$

which is $\gg q^\varepsilon$ because we are operating under the conditions of Lemma 8.5.

Now we observe that the integrand is very small unless

$$\frac{Na M_1}{CK} |1 \pm \cos \theta| \ll q^\varepsilon.$$

Hence, the sign must be $-$, and we must have

$$\theta \ll \left(\frac{CK}{Na M_1} \right)^{1/2} q^\varepsilon,$$

(which is $O(q^{-\delta})$, for some $\delta > 0$). This means that by using a Taylor expansion, we may develop the $\widehat{w_T}$ part into an asymptotic expansion with leading term given by the substitution $1 - \cos \theta \rightarrow \theta^2/2$. Therefore,

$$Z = \frac{K}{g_0 k_1} \int_{-\infty}^\infty \cos((\ell-1)\theta) (\widehat{w_T} \left(\frac{z g_0 k_1}{K} \theta^2 \right) + \dots) d\theta,$$

where we were able to extend the integral to $+\infty$ since $\widehat{w_T}$ is small otherwise, and also extend to $-\infty$ by symmetry (we have also re-defined the inert function to absorb constants).

As another shorthand, let

$$Q = \frac{zg_0k_1}{K} \asymp \frac{NaM_1}{CK}.$$

Then Z takes the form

$$Z = \frac{K}{g_0k_1\sqrt{Q}} \int_{-\infty}^{\infty} \exp\left(i\frac{(\ell-1)}{\sqrt{Q}}\theta\right) \widehat{w}_T(\theta^2) d\theta + \dots$$

If we let $g(\theta) = \widehat{w}_T(\theta^2)$, then $g^{(j)}(\theta) \ll_{j,A} X^j(1+\theta)^{-A}$, for arbitrary j, A , where $X \ll q^\varepsilon$. Therefore, this is another Fourier transform of a function with controlled derivatives, and it is not hard to see that it takes the form

$$\frac{K}{g_0k_1\sqrt{Q}} G\left(\frac{\ell-1}{\sqrt{Q}}, \cdot\right),$$

plus a very small error term, where G would be q^ε -inert (in ℓ) if it had dyadic support. It is q^ε -inert in all the other variables, however.

Re-grouping, we have that

$$W_h = \left(\frac{\sqrt{aMN}}{C}\right)^{\kappa-1} \sqrt{M_2N} \left(\frac{CK}{aM_1N}\right)^2 K \frac{\sqrt{\delta_2}}{fg_0k_1h} G\left(\frac{\ell-1}{\sqrt{Q}}, \cdot\right),$$

plus a very small error term, where G is very small unless

$$\ell \ll \left(\frac{M_1aN}{CK}\right)^{1/2} q^\varepsilon.$$

Then we may take the Mellin transform in p_1, p_2, p_3, c_0 , giving

$$(10.6) \quad W_h(p_1, p_2, p_3; c'_0; \ell) = \left(\frac{\sqrt{aMN}}{C}\right)^{\kappa-1} \sqrt{M_2N} \left(\frac{CK}{aM_1N}\right)^2 K \frac{\sqrt{\delta_2}}{fg_0k_1h} \int_{(\sigma)} \widetilde{w}_T(\mathbf{u}, \ell) \left(\frac{P_1}{|p_1|}\right)^{u_1} \left(\frac{P_2}{p_2}\right)^{u_2} \left(\frac{P_3}{p_3}\right)^{u_3} \left(\frac{C}{\delta_4 c'_0 c_2}\right)^{u_4} d\mathbf{u},$$

plus a small error term.

Now we turn to W_+ . Since the details are similar to the previous case, the exposition is brief. We follow through the steps used for W_h , where the alteration in the first step is replacing $J_{\ell-1}$ by $B_{2ir}^+(x)$. In place of (10.5), we have instead

$$\frac{J_{2ir}(x) - J_{-2ir}(x)}{\sinh(\pi r)} = \frac{2}{\pi i} \int_{-\infty}^{\infty} \cos(x \cosh v) e\left(\frac{rv}{\pi}\right) dv.$$

We shall use this for real values of r . Forming the analog of Z from W_h , and keeping the same definition of z , we have (absorbing the absolute constant into the inert function)

$$Z = \int_{-\infty}^{\infty} e\left(\frac{rv}{\pi}\right) \int_{-\infty}^{\infty} e\left(\frac{z}{y}\right) \cos\left(2\pi \frac{z}{y} \cosh(v)\right) w_T(y, \cdot) dy dv.$$

Next write $\cos(u) = \frac{1}{2}e^{iu} + \frac{1}{2}e^{-iu}$; the part with e^{iu} is very small as in the W_h case. From this point on, the analysis is nearly identical to that of W_h , and the conclusion is that W_+ is very small unless

$$|t_j| \ll \left(\frac{M_1aN}{CK}\right)^{1/2} q^\varepsilon,$$

and W_+ satisfies a formula identical to that in (10.6).

In the exceptional eigenvalue case where $ir \in \mathbb{R}$, then the final shape of the formula for W_+ is the same as (10.6), but the above arguments would need modification since $e(rv/\pi)$ is no longer bounded. There is a more direct route, however. We have the asymptotic expansion (see [14, (8.451.1)])

$$\frac{J_{2ir}(x) - J_{-2ir}(x)}{\sinh(\pi r)} \sim \sum_{\pm} e^{\pm ix} \sum_k \frac{P_{\pm}(r, k)}{x^{\frac{1}{2}+k}},$$

where $P_{\pm}(r, k)$ is a polynomial in r and k . This is certainly valid for $r = O(1)$, and $x \gg 1$ (in the present context, $x \gg q^\varepsilon$). With this, it is easy to estimate Z directly, showing that it is of the form $\frac{K}{g_0 k_1 \sqrt{Q}}$ times an X -inert function, plus a small error term. Therefore, applying Mellin inversion in the appropriate variables, we obtain an expression of the same form as (10.6).

11. REGROUPING AFTER BRUGGEMAN-KUZNETSOV

11.1. Non-oscillatory, Maass cases. Here we consider the contribution to \mathcal{T}_d from the parameters where B is non-oscillatory. By (9.4), (9.6), and (10.3), we obtain

$$(11.1) \quad \mathcal{T}_d^{\pm} = \frac{g_0 k_1}{K} \sum_{\delta_3 | \frac{c_2}{(g_0, c_2)}} \sum_{(\delta_4, \delta_2 g_0 m'_1)=1} \frac{\mu(\delta_4)}{\delta_4} \sum_{\substack{(f, \delta_2 \delta_4)=1 \\ (9.1) \text{ is true}}} \sum_{\substack{(c'_0, f g_0 m'_1)=1 \\ \delta_4 c'_0 \equiv 0 \pmod{q k_1 k_1^*}}} \frac{1}{c'_0} \sum_{\pm p_1, p'_2, p'_3 \geq 1} \\ \sum_{t_j \text{ level } \delta_2 \delta_5} \nu_{\infty, j}(p_1 \delta_4 c'_0) \bar{\nu}_{\frac{1}{\delta_5}, j}(p'_2 p'_3) \left(\frac{\sqrt{aMN}}{C} \right)^{\delta} (M_2 N)^{1/2} \frac{\sqrt{\delta_2}}{h} \int_{(2\theta+\varepsilon)} h_{\pm}(s, t_j) \\ \int_{(\sigma)} \left(\frac{\sqrt{\delta_2} K}{f g_0 k_1 \sqrt{\delta_4 c'_0 |p_1| p'_2 p'_3}} \right)^s \widetilde{w}_T(s, \mathbf{u}) \left(\frac{P_1}{|p_1|} \right)^{u_1} \left(\frac{P_2}{f p'_2} \right)^{u_2} \left(\frac{P_3}{f p'_3} \right)^{u_3} \left(\frac{C}{\delta_4 c'_0 c_2} \right)^{u_4} d\mathbf{u} ds,$$

plus a very small error term. In the above expression, we could take $\text{Re}(s) > 2\theta$ without crossing any poles coming from exceptional Laplace eigenvalues (recall (4.8) for the definition of h_{\pm}). By Lemma 10.1, we may truncate at $|t_j| \ll (1+Y)q^\varepsilon$ with a small error term. Now we move the sums over p_1, p'_2, p'_3 , and c_0 to the inside, change variables $u_i \rightarrow u_i - \frac{s}{2}$, and bound everything at that point with absolute values. In this way, we obtain

$$(11.2) \quad \mathcal{T}_d^{\pm} \ll q^\varepsilon \frac{g_0 k_1 c_2}{KC} \left(\frac{\sqrt{aMN}}{C} \right)^{\delta} (M_2 N)^{1/2} \frac{\sqrt{\delta_2}}{h} \sum_{\delta_3 | \frac{c_2}{(g_0, c_2)}} \sum_{(\delta_4, \delta_2 g_0 m'_1)=1} \sum_{\substack{(f, \delta_2 \delta_4)=1 \\ (9.1) \text{ is true}}} \sum_{t_j \text{ level } \delta_2 \delta_5} \\ \frac{1}{1+|t_j|} \int_{(2\theta+\varepsilon)} \int_{(\sigma)} \left(\frac{t_j}{Y} \right)^{2\theta+\varepsilon} |\widetilde{w}_T(s, \mathbf{u} - \frac{s}{2})| \left| P_1^{u_1} \left(\frac{P_2}{f} \right)^{u_2} \left(\frac{P_3}{f} \right)^{u_3} \left(\frac{C}{\delta_4 c_2} \right)^{u_4} \right| |Z_j(\mathbf{u})| d\mathbf{u} ds,$$

plus a very small error term, where

$$(11.3) \quad Z_j(\mathbf{u}) = \sum_{\substack{(c'_0, f g_0 m'_1)=1 \\ \delta_4 c'_0 \equiv 0 \pmod{q k_1 k_1^*}}} \sum_{p_1, p'_2, p'_3 \geq 1} \frac{\nu_{\infty, j}(p_1 \delta_4 c'_0) \bar{\nu}_{\frac{1}{\delta_5}, j}(p'_2 p'_3)}{p_1^{u_1} p_2^{u_2} p_3^{u_3} c_0^{u_4}}.$$

Our plan is to relate $Z_j(\mathbf{u})$ to L -functions, and use a large sieve inequality to bound it on average over t_j .

11.2. Non-oscillatory, Holomorphic cases. These cases are nearly identical to those in Section 11.1, but the bounds will turn out to be even better due to the applicability of Deligne's bound. The key point is that for $k \ll (1+Y)q^\varepsilon$, we may claim the bound

$$|h(s, k)| \ll k^{\sigma-1},$$

which is entirely analogous to $|h(s, t_j)| \ll t_j^{\sigma-1}$. We omit the details for brevity.

11.3. Oscillatory, Maass cases. As in Section 11.1, we use (9.4) and (9.6), but instead of (10.3) we use a variant on (10.6). Also recall that only the $+$ sign enters the picture in the oscillatory case. Thus we obtain

$$(11.4) \quad \mathcal{T}_d^+ \ll \frac{g_0 k_1 c_2}{KC} \left(\frac{\sqrt{aMN}}{C} \right)^\delta (M_2 N)^{1/2} \frac{\sqrt{\delta_2}}{h} \sum_{\delta_3 | \frac{c_2}{(g_0, c_2)}} \sum_{(\delta_4, \delta_2 g_0 m'_1)=1} \sum_{\substack{(f, \delta_2 \delta_4 c'_0)=1 \\ (9.1) \text{ is true}}} \sum_{\substack{t_j \text{ level } \delta_2 \delta_5 \\ |t_j| \ll Y' q^\varepsilon}} \left(\frac{CK}{aM_1 N} \right)^2 \int_{(\sigma)} |\widetilde{w}_T(\mathbf{u}, t_j)| \left| P_1^{u_1} \left(\frac{P_2}{f} \right)^{u_2} \left(\frac{P_3}{f} \right)^{u_3} \left(\frac{C}{\delta_4 c_2} \right)^{u_4} \right| |Z_j(\mathbf{u})| d\mathbf{u},$$

where

$$(11.5) \quad Y' = \left(\frac{M_1 a N}{CK} \right)^{1/2} q^\varepsilon.$$

11.4. Oscillatory, Holomorphic cases. These are similar to (but easier than) the Oscillatory, Maass cases, and so we omit them.

11.5. Continuous spectrum. First consider the non-oscillatory cases. Then analogously to (11.1), we have

$$(11.6) \quad \mathcal{T}_c^\pm = \frac{g_0 k_1}{K} \sum_{\delta_3 | \frac{c_2}{(g_0, c_2)}} \sum_{(\delta_4, \delta_2 g_0 m'_1)=1} \frac{\mu(\delta_4)}{\delta_4} \sum_{\substack{(f, \delta_2 \delta_4)=1 \\ (9.1) \text{ is true}}} \sum_{\substack{(c'_0, f g_0 m'_1)=1 \\ \delta_4 c'_0 \equiv 0 \pmod{q k_1 k_1^*}}} \frac{1}{c'_0} \sum_{\pm p_1, p'_2, p'_3 \geq 1} \sum_{\mathfrak{c}} \int_t \nu_{\infty, \mathfrak{c}}(p_1 \delta_4 c'_0, \frac{1}{2} + it) \overline{\nu}_{\frac{1}{\delta_5}, \mathfrak{c}}(p'_2 p'_3, \frac{1}{2} + it) \left(\frac{\sqrt{aMN}}{C} \right)^\delta (M_2 N)^{1/2} \frac{\sqrt{\delta_2}}{h} \int_{(\varepsilon)} h_\pm(s, t) \int_{(1+\varepsilon)} \left(\frac{\sqrt{\delta_2} K}{f g_0 k_1 \sqrt{\delta_4 c'_0 |p_1| p'_2 p'_3}} \right)^s \widetilde{w}_T(s, \mathbf{u}) \left(\frac{P_1}{|p_1|} \right)^{u_1} \left(\frac{P_2}{f p'_2} \right)^{u_2} \left(\frac{P_3}{f p'_3} \right)^{u_3} \left(\frac{C}{\delta_4 c'_0 c_2} \right)^{u_4} d\mathbf{u} ds dt.$$

Now we move the sums to the inside, getting

$$(11.7) \quad \mathcal{T}_c^\pm \ll q^\varepsilon \frac{g_0 k_1 c_2}{KC} \left(\frac{\sqrt{aMN}}{C} \right)^\delta (M_2 N)^{1/2} \frac{\sqrt{\delta_2}}{h} \sum_{\delta_3 | \frac{c_2}{(g_0, c_2)}} \sum_{(\delta_4, \delta_2 g_0 m'_1)=1} \sum_{\substack{(f, \delta_2 \delta_4)=1 \\ (9.1) \text{ is true}}} \int_{|t| \ll (1+Y)q^\varepsilon} \frac{1}{1+|t|} \int_{(\varepsilon)} \int_{(1+\varepsilon)} |\widetilde{w}_T(s, \mathbf{u} - \frac{s}{2})| \left| P_1^{u_1} \left(\frac{P_2}{f} \right)^{u_2} \left(\frac{P_3}{f} \right)^{u_3} \left(\frac{C}{\delta_4 c_2} \right)^{u_4} \right| \sum_{\mathfrak{c}} |Z_{\mathfrak{c}, t}(\mathbf{u})| d\mathbf{u} ds,$$

where

$$(11.8) \quad Z_{c,t}(\mathbf{u}) = \sum_{\substack{(c'_0, fg_0 m'_1)=1 \\ \delta_4 c'_0 \equiv 0 \pmod{q k_1 k_1^*}}} \sum_{p_1, p'_2, p'_3 \geq 1} \frac{\nu_{\infty, c}(p_1 \delta_4 c'_0, \frac{1}{2} + it) \overline{\nu}_{\frac{1}{\delta_5}, c}(p'_2 p'_3, \frac{1}{2} + it)}{p_1^{u_1} p_2^{u_2} p_3^{u_3} c_0^{u_4}}.$$

The **oscillatory case** is similar, leading to

$$(11.9) \quad \mathcal{T}_c^+ \ll \frac{g_0 k_1 c_2}{KC} \left(\frac{\sqrt{aMN}}{C} \right)^{\kappa-1} (M_2 N)^{1/2} \frac{\sqrt{\delta_2}}{h} \sum_{\delta_3 | \frac{c_2}{(g_0, c_2)}} \sum_{(\delta_4, \delta_2 g_0 m'_1)=1} \sum_{\substack{(f, \delta_2 \delta_4 c'_0)=1 \\ (9.1) \text{ is true}}} \int_{|t| \ll Y' q^\varepsilon} \left(\frac{CK}{aM_1 N} \right)^2 \int_{(1+\varepsilon)} |\widetilde{w}_T(\mathbf{u}, t)| \left| P_1^{u_1} \left(\frac{P_2}{f} \right)^{u_2} \left(\frac{P_3}{f} \right)^{u_3} \left(\frac{C}{\delta_4 c_2} \right)^{u_4} \right| \sum_{\mathbf{c}} |Z_{c,t}(\mathbf{u})| d\mathbf{u} dt.$$

11.6. Claiming bounds on Z_j , and estimating \mathcal{T} . In Section 12 below, we will show the following

Lemma 11.1. *The function $Z_j(\mathbf{u})$ has analytic continuation to $\text{Re}(\mathbf{u}) \geq \sigma > 1/2$. In this region, and assuming $|\text{Im}(\mathbf{u})| \ll q^\varepsilon$, it satisfies the bound*

$$(11.10) \quad \sum_{\substack{t_j \text{ level } \delta_2 \delta_5 \\ |t_j| \leq T}} |Z_j(\mathbf{u})| \ll_{\sigma, \varepsilon} q^{\theta - \frac{1}{2}} \frac{(\delta_4, q)^{1/2}}{(k_1 k_1^*)^{\frac{1}{2}} \delta_4^{1/2}} T^{2+\varepsilon} q^\varepsilon.$$

The key feature is that this bound saves a factor $\delta_4^{1/2}$ which ultimately arises from (4.49).

Now we use Lemma 11.1 to estimate \mathcal{T}_d^\pm , and eventually \mathcal{S} . We do not require the factor $(k_1 k_1^*)^{-1/2}$ appearing in (11.10), and in order to unify the treatment with the continuous spectrum, we shall use the weaker bound without this factor.

First consider the **Non-oscillatory, Maass cases**. Inserting the bound from Lemma 11.1 into (11.2) (taking $\sigma = 1/2 + \varepsilon$), and trivially summing over δ_4 (here is where the savings of $\delta_4^{1/2}$ is important), f , and δ_3 , we obtain

$$(11.11) \quad \mathcal{T}_d^\pm \ll q^\varepsilon \frac{g_0 k_1 c_2}{KC} \left(\frac{\sqrt{aMN}}{C} \right)^\delta (M_2 N)^{1/2} \frac{\sqrt{\delta_2}}{h} (Y^{-2\theta} + Y) P^{1/2} \left(\frac{C}{c_2} \right)^{1/2} q^{\theta - \frac{1}{2}}.$$

Let us call \mathcal{S}_d^\pm for the contribution to \mathcal{S} from this part. Applying the additional summations that led from \mathcal{S} to \mathcal{S}''' (see (7.10), (7.6), (7.1)), we obtain

$$\begin{aligned} \mathcal{S}_d^\pm &\ll q^\varepsilon \sum_a \frac{1}{a^{3/2}} \sum_{c_2} \frac{1}{c_2^{3/2}} \sum_{d|c_2} d \sum_{k_1} k_1^{1/2} \sum_{m'_1} \frac{1}{\sqrt{m'_1}} \sum_{r_1 r_2 r_3 = \delta_1} \sum_{e_1 | r_2 r_3} \sum_{e_2 | r_3} \sum_{\substack{g_0 | e_1 e_2 \delta_1 a m'_1 \\ g_0 \equiv 0 \pmod{d}}} \sum \\ &\quad \times \frac{g_0 k_1 c_2}{KC} \left(\frac{\sqrt{aMN}}{C} \right)^\delta (M_2 N)^{1/2} \frac{\sqrt{\delta_2}}{h} (Y^{-2\theta} + Y) P^{1/2} \left(\frac{C}{c_2} \right)^{1/2} q^{\theta - \frac{1}{2}}. \end{aligned}$$

Convention. Here and below, we have not written the truncation points for these outer summation variables. In almost all cases, all that is necessary is to recall that all the variables may be bounded by some fixed power of q . The only exception is that for some estimates we need to use that $m'_1 \ll \frac{M_1}{k_1 c_2}$.

For convenience, we gather some of the previous definitions:

$$(11.12) \quad \begin{aligned} h &= e_1 e_2 r_1 r_2 r_3, \quad \delta_1 = r_1 r_2 r_3 = \frac{k_1 d}{(a, k_1 d)}, \quad \delta_2 = \frac{e_1 e_2 \delta_1 a m'_1}{g_0} = \frac{h a m'_1}{g_0}, \\ N' h &= N, \quad m_1 = k_1 c_2 m'_1, \quad Y = \frac{g_0 k_1 \sqrt{CP}}{\sqrt{\delta_2 c_2 K}}. \end{aligned}$$

With these substitutions, we obtain

$$\sqrt{\delta_2} (Y^{-2\theta} + Y) P^{1/2} \left(\frac{C}{c_2} \right)^{1/2} = \frac{\delta_2 K}{g_0 k_1} (Y^{1-2\theta} + Y^2),$$

and hence

$$(11.13) \quad \begin{aligned} \mathcal{S}_d^\pm &\ll q^\varepsilon \sum_a \frac{1}{a^{3/2}} \sum_{c_2} \frac{1}{c_2^{3/2}} \sum_{d|c_2} d \sum_{k_1} k_1^{1/2} \sum_{m'_1} \frac{1}{\sqrt{m'_1}} \sum_{r_1 r_2 r_3 = \delta_1} \sum_{e_1 | r_2 r_3} \sum_{e_2 | r_3} \sum_{\substack{g_0 | e_1 e_2 \delta_1 a m'_1 \\ g_0 \equiv 0 \pmod{d}}} \\ &\quad \times \left(\frac{\sqrt{aMN}}{C} \right)^\delta M_2^{1/2} N^{1/2} \frac{c_2}{C} \frac{a m'_1}{g_0} (Y^{1-2\theta} + Y^2) q^{\theta - \frac{1}{2}}. \end{aligned}$$

Now we note that in this non-oscillatory case, we have from (8.7) that

$$\frac{P(g_0 k_1)^2}{\delta_2 c_2} \ll q^\varepsilon \frac{K^3}{NaM_1},$$

which in particular means that $Y \ll (\frac{CK}{NaM_1})^{1/2} q^\varepsilon$, which is independent of g_0, k_1, c_2 , etc. Now it is evident that the sums over $g_0, e_1, e_2, r_1, r_2, r_3$ contribute at most $O(q^\varepsilon)$, and the fact that $d|g_0$ will cancel the other visible factor of d in (11.13). With this observation, and performing minor simplifications, we have

$$\begin{aligned} \mathcal{S}_d^\pm &\ll q^\varepsilon \sum_a \frac{1}{a^{1/2}} \sum_{c_2} \frac{1}{c_2^{1/2}} \sum_{d|c_2} \sum_{k_1} k_1^{1/2} \sum_{m'_1} \sqrt{m'_1} \\ &\quad \times \left(\frac{\sqrt{aMN}}{C} \right)^\delta \frac{M_2^{1/2} N^{1/2}}{C} q^{\theta - \frac{1}{2}} \left(\left(\frac{\sqrt{CK}}{\sqrt{NaM_1}} \right)^{1-2\theta} + \left(\frac{\sqrt{CK}}{\sqrt{NaM_1}} \right)^2 \right). \end{aligned}$$

Trivially summing over m'_1 (recall $m'_1 \ll \frac{M_1}{k_1 c_2}$), k_1, d, c_2 , and finally a , we derive

$$\mathcal{S}_d^\pm \ll q^\varepsilon \max_a M_1^{3/2} \left(\frac{\sqrt{aMN}}{C} \right)^\delta \frac{M_2^{1/2} N^{1/2} a^{1/2}}{C} q^{\theta - \frac{1}{2}} \left(\left(\frac{\sqrt{CK}}{\sqrt{NaM_1}} \right)^{1-2\theta} + \left(\frac{\sqrt{CK}}{\sqrt{NaM_1}} \right)^2 \right).$$

Now we split it up into the cases from Lemmas 8.3 and 8.4. In the **case of Lemma 8.3**, we have $\delta = -1$, and

$$M_1^{3/2} \left(\frac{\sqrt{aMN}}{C} \right)^\delta \frac{M_2^{1/2} N^{1/2} a^{1/2}}{C} q^{\theta - \frac{1}{2}} = \frac{M_1}{q^{\frac{1}{2} - \theta}}.$$

Meanwhile, using $K \asymp M_2^{-1} \sqrt{aMN}$ (see (6.23)), we have

$$\frac{CK}{NaM_1} \asymp \frac{C}{\sqrt{aMN}} \ll q^\varepsilon.$$

via (6.21). Therefore, in this case we have

$$(11.14) \quad \mathcal{S}_d^\pm \ll \frac{M_1}{q^{1/2}} q^{\theta+\varepsilon} \ll q^{\theta+\varepsilon}.$$

In the **case of Lemma 8.4**, we have $\frac{CK}{NaM_1} \gg q^{-\varepsilon}$, and $\delta = \kappa - 1 \geq 1$, so with easy simplifications, we derive

$$\mathcal{S}_d^\pm \ll q^\varepsilon \max_a \frac{M_2 K}{C} \frac{M_1 q^\theta}{q^{\frac{1}{2}}}.$$

Since $\frac{KM_2}{C} \ll q^\varepsilon$ in this case, we obtain the same bound as (11.14).

The Non-oscillatory, holomorphic cases are nearly identical, so we omit the proofs.

Now consider the **oscillatory, Maass case**, where we treat (11.4). Following the same steps as the non-oscillatory cases, we obtain

$$\mathcal{T}_d^+ \ll \frac{g_0 k_1}{K} \left(\frac{\sqrt{aMN}}{C} \right)^\delta (M_2 N)^{1/2} \frac{\sqrt{\delta_2}}{h} \left(\frac{CK}{aM_1 N} \right)^{P^{1/2}} \left(\frac{c_2}{C} \right)^{1/2} \frac{q^\theta}{q^{1/2}}.$$

After some simplifications, we have

$$(11.15) \quad \mathcal{S}_d^+ \ll q^\varepsilon \sum_a \frac{1}{a^{3/2}} \sum_{c_2} \frac{1}{c_2^{3/2}} \sum_{d|c_2} d \sum_{k_1} k_1^{1/2} \sum_{m'_1} \frac{1}{\sqrt{m'_1}} \sum_{r_1 r_2 r_3 = \delta_1} \sum_{e_1 | r_2 r_3} \sum_{e_2 | r_3} \sum_{\substack{g_0 | e_1 e_2 \delta_1 a m'_1 \\ g_0 \equiv 0 \pmod{d}}} \frac{g_0 k_1}{K} \left(\frac{\sqrt{aMN}}{C} \right)^\delta (M_2 N)^{1/2} \left(\frac{c_2}{C} \right)^{1/2} \frac{\sqrt{\delta_2}}{h} \left(\frac{CK}{aM_1 N} \right)^{P^{1/2}} \frac{q^\theta}{q^{1/2}}.$$

We need to remember the origins of these variables. We have

$$(11.16) \quad P = P_1 P_2 P_3 \asymp \frac{(NaM_1)^3 k_0'^3}{C^3 K^3 N'} \asymp \left(\frac{NaM_1}{CK} \right)^3 \frac{K^3}{N} \frac{h}{(g_0 k_1)^3}.$$

Thus the bound becomes

$$\mathcal{S}_d^+ \ll q^\varepsilon \sum_a \frac{1}{a^{3/2}} \sum_{c_2} \frac{1}{c_2^{3/2}} \sum_{d|c_2} d \sum_{k_1} k_1^{1/2} \sum_{m'_1} \frac{1}{\sqrt{m'_1}} \sum_{r_1 r_2 r_3 = \delta_1} \sum_{e_1 | r_2 r_3} \sum_{e_2 | r_3} \sum_{\substack{g_0 | e_1 e_2 \delta_1 a m'_1 \\ g_0 \equiv 0 \pmod{d}}} \frac{g_0 k_1}{K} \left(\frac{\sqrt{aMN}}{C} \right)^\delta (M_2 N)^{1/2} \left(\frac{c_2}{C} \right)^{1/2} \sqrt{\frac{h a m'_1}{g_0}} \left(\frac{NaM_1}{CK} \right)^{1/2} \left(\frac{K^3}{N} \frac{h}{(g_0 k_1)^3} \right)^{1/2} \frac{q^\theta}{h q^{1/2}}.$$

We see that the sum over g_0 gives $O(d^{-1} q^\varepsilon)$, and the h -dependence cancels out entirely, so that the δ_1 -dependence is also essentially gone. Thus, we obtain

$$\mathcal{S}_d^+ \ll q^\varepsilon \frac{q^\theta}{q^{1/2}} \sum_a \frac{1}{a} \sum_{c_2} \frac{1}{c_2} \sum_{d|c_2} \sum_{k_1} (k_1 k_1^*)^{-1/2} \sum_{m'_1} \frac{(M_2 N)^{1/2}}{K C^{1/2}} \left(\frac{\sqrt{aMN}}{C} \right)^\delta \left(\frac{NaM_1}{CK} \right)^{1/2} \left(\frac{K^3}{N} \right)^{1/2}.$$

Now we sum over all the remaining variables, giving in all

$$\mathcal{S}_d^+ \ll q^\varepsilon \frac{M_1 q^\theta}{q^{1/2}} \frac{(M_2 N)^{1/2}}{K C^{1/2}} \left(\frac{\sqrt{aMN}}{C} \right)^\delta \left(\frac{NaM_1}{CK} \right)^{1/2} \left(\frac{K^3}{N} \right)^{1/2}.$$

Simplifying (in particular, $\delta = \kappa - 1$ here), we obtain

$$\mathcal{S}_d^+ \ll q^\varepsilon \frac{M_1 q^\theta}{q^{1/2}} \frac{MaN}{C^2}.$$

Since $\sqrt{MaN} \ll Cq^\varepsilon$ (see (6.18)), we obtain the same bound as (11.14). The **oscillatory, holomorphic case** is similar, but even simpler.

In summary, this shows the desired bound for the Maass forms and holomorphic forms.

11.7. Claiming bounds on $Z_{c,t}$, and estimating \mathcal{T}_c . Recall the definition (11.8). Define flrt to be the multiplicative function defined on prime powers by

$$(11.17) \quad \text{flrt}(p^\alpha) = p^{\lfloor \alpha/2 \rfloor}.$$

Lemma 11.2. *The function $Z_{c,t}(\mathbf{u})$ has a decomposition of the following form. We have*

$$Z_{c,t}(u_1, u_2, u_3, u_4) = (Z_1^0(u_2, u_3) + Z_1^*(u_2, u_3))(Z_2^0(u_1, u_4) + Z_2^*(u_1, u_4)),$$

where for $i = 1, 2$, $Z_i^*(\alpha, \beta)$ has analytic continuation to $\text{Re}(\alpha, \beta) \geq \sigma > 1/2$, and $Z_i^0(\alpha, \beta)$ is analytic for $\text{Re}(\alpha, \beta) \geq \sigma > 1$. For the four bounds stated below, assume $|\text{Im}(u_i)| \ll q^\varepsilon$ for $i = 1, 2, 3, 4$. For $\text{Re}(\mathbf{u}) \geq \sigma > 1/2$, we have

$$(11.18) \quad \int_{|t| \leq T} \sum_{\mathfrak{c}} |Z_1^*(u_2, u_3) Z_2^*(u_1, u_4)| dt \ll q^\varepsilon T^{2+\varepsilon} \left(\frac{(\delta_4, q)}{q} \right)^{1/2} \frac{\text{flrt}(\delta_2) \text{flrt}(\delta_3)^{3/2}}{\sqrt{\delta_2 \delta_5}}.$$

For $\text{Re}(\mathbf{u}) \geq \sigma > 1$, we have

$$(11.19) \quad \sum_{\mathfrak{c}} |Z_1^0(u_2, u_3) Z_2^0(u_1, u_4)| \ll (q(1 + |t|))^\varepsilon \frac{(\delta_4, q)}{q \sqrt{k_1 k_1^*}} \frac{1}{\delta_2 \delta_5}.$$

For $\text{Re}(u_1, u_4) \geq \sigma > 1$ and $\text{Re}(u_2, u_3) \geq \sigma' > 1/2$, we have

$$(11.20) \quad \int_{|t| \leq T} \sum_{\mathfrak{c}} |Z_1^*(u_2, u_3) Z_2^0(u_1, u_4)| dt \ll q^\varepsilon T^{1+\varepsilon} \frac{(\delta_4, q)}{q \sqrt{k_1 k_1^*}} \frac{\text{flrt}(\delta_2) \sqrt{\text{flrt}(\delta_3)}}{\delta_2 \sqrt{\delta_5}}.$$

For $\text{Re}(u_2, u_3) \geq \sigma > 1$ and $\text{Re}(u_1, u_4) \geq \sigma' > 1/2$, we have

$$(11.21) \quad \int_{|t| \leq T} \sum_{\mathfrak{c}} |Z_1^0(u_2, u_3) Z_2^*(u_1, u_4)| dt \ll q^\varepsilon T^{1+\varepsilon} \left(\frac{(\delta_4, q)}{q} \right)^{1/2} \frac{\text{flrt}(\delta_2) \text{flrt}(\delta_3)}{\delta_2 \delta_5}.$$

We remark on some important features of the above bounds. In (11.18) and (11.21) we require a factor $\delta_5^{-1/2}$ (or better) to secure convergence of the sum over δ_4 . The overall power of k_1 is also important for securing convergence in each case. In terms of the final power of q that occurs in our bound on \mathcal{S}_c , the most important feature is the power of δ_2 . This is because δ_2 contains the m'_1 variable which can be as large as $q^{1/2+\varepsilon}$. Note that although $\text{flrt}(n)$ may occasionally be as large as \sqrt{n} , it is small on average, indeed $\sum_{n \leq x} \text{flrt}(n) \ll x \log x$.

Using Lemma 11.2, we bound \mathcal{T}_c . For the non-oscillatory cases, we return to (11.7). Technically, we should return to (11.6), decompose \mathcal{T}_c^\pm according to $Z_{c,t} = (Z_1^* + Z_1^0)(Z_2^* + Z_2^0)$ into four pieces, shift contours appropriately, and only then apply the absolute values. We found it slightly easier to bound Z_1^0 and Z_2^0 slightly to the right of the 1-line instead of bounding the residues of Z_1 and Z_2 , but this is more-or-less equivalent.

Then (note that the sum over f converges absolutely, and the t -integral is easily estimated, so we may simplify a bit in these aspects)

$$\begin{aligned} \mathcal{T}_c^\pm \ll q^\varepsilon \frac{g_0 k_1 c_2}{KC} \left(\frac{\sqrt{aMN}}{C} \right)^\delta (M_2 N)^{1/2} \frac{\sqrt{\delta_2}}{h} \sum_{\delta_3 | \frac{c_2}{(g_0, c_2)}} \sum_{(\delta_4, \delta_2 g_0 m'_1)=1} \text{flrt}(\delta_2) \text{flrt}(\delta_3) \\ \left(\frac{(\delta_4, q)^{1/2}}{q^{1/2}} (1+Y) \frac{(P_1 P_2 P_3 C)^{1/2} \text{flrt}(\delta_3)^{1/2}}{\sqrt{\delta_2 \delta_4 \delta_5 c_2}} + \frac{(\delta_4, q)}{q} \frac{P_1 P_2 P_3 C}{\delta_2 \delta_4 \delta_5 c_2 \sqrt{k_1 k_1^*}} \right. \\ \left. + \frac{P_1 (P_2 P_3)^{1/2} C}{\delta_2 \delta_4 c_2 \sqrt{\delta_5}} \frac{(\delta_4, q)}{q \sqrt{k_1 k_1^*}} + \frac{P_1^{1/2} P_2 P_3 C^{1/2}}{\delta_4^{1/2} c_2^{1/2} \delta_2 \delta_5} \frac{(\delta_4, q)^{1/2}}{q^{1/2}} \right). \end{aligned}$$

Using $\delta_5 \geq \sqrt{\delta_3 \delta_4}$ (recall that $\delta_5 = [\delta_3, \delta_4]$), the sums over δ_3 and δ_4 are easily evaluated, and lead to a factor of size at most $O(q^\varepsilon)$; the only slightly tricky case uses instead

$$(11.22) \quad \text{flrt}(\delta_3)^{1/2} \sum_{\delta_4} \frac{(\delta_4, q)^{1/2}}{\sqrt{\delta_4 \delta_5}} = \text{flrt}(\delta_3)^{1/2} \sum_{\delta_4} \frac{(\delta_4, \delta_3 q)^{1/2}}{\delta_4 \sqrt{\delta_3}} \ll \text{flrt}(c_2) \frac{(\delta_3 q)^\varepsilon}{\sqrt{\delta_3}}.$$

It is helpful to observe the following nice simplification. At this point we can see that the first term within the parentheses which occurred from $Z_1^* Z_2^*$ will lead to the same bound we obtained on \mathcal{T}_d^\pm , by comparison to (11.11). The only difference is the benign factor of $\text{flrt}(c_2)$, which does not make the sum over c_2 appreciably larger, since $\sum_{c_2} c_2^{-1} \text{flrt}(c_2) \ll q^\varepsilon$. Actually, apart from $\text{flrt}(c_2)$, the bound is better in two ways: firstly, the factor q^θ may be omitted, and secondly, instead of using $\frac{\text{flrt}(\delta_2)}{\sqrt{\delta_2}} \leq 1$, we could use that $\text{flrt}(n)$ is $O(n^\varepsilon)$ on average, which could lead to a saving of the factor $M_1^{1/2}$. Instead of carrying through the calculations, we will simply abbreviate this term by $(**)$ in the forthcoming calculations.

Next we wish to sum over the outer variables that make \mathcal{S}_c from \mathcal{T}_c . To this end, we need to write the P_i , Y , and δ_2 variables in terms of these outer ones. Let $P_i^* = \frac{K}{N_i}$, so that $P_i \ll q^\varepsilon P_i^* \frac{h_i}{g_0 k_1}$ where $h_1 = e_1 r_1$, $h_2 = e_2 r_2$, and $h_3 = r_3$ (so $h = h_1 h_2 h_3$). With this, we obtain

$$\begin{aligned} \mathcal{T}_c^\pm \ll q^\varepsilon \frac{g_0 k_1 c_2}{KC} \left(\frac{\sqrt{aMN}}{C} \right)^\delta (M_2 N)^{1/2} \frac{\sqrt{\delta_2}}{h} \text{flrt}(\delta_2) \\ \left[(**) + \frac{h}{(g_0 k_1)^3} \frac{P_1^* P_2^* P_3^* C}{q \delta_2 c_2 \sqrt{k_1 k_1^*}} + \frac{h_1 (h_2 h_3)^{1/2}}{(g_0 k_1)^2} \frac{P_1^* (P_2^* P_3^*)^{1/2} C}{q \delta_2 c_2 \sqrt{k_1 k_1^*}} + \frac{h_1^{1/2} h_2 h_3}{(g_0 k_1)^{5/2}} \frac{(P_1^*)^{1/2} P_2^* P_3^* C^{1/2}}{q^{1/2} c_2^{1/2} \delta_2} \right]. \end{aligned}$$

Recall that

$$\mathcal{S}_c^\pm \ll q^\varepsilon \sum_a \frac{1}{a^{3/2}} \sum_{c_2} \frac{1}{c_2^{3/2}} \sum_{d|c_2} d \sum_{k_1} k_1^{1/2} \sum_{m'_1} \frac{1}{\sqrt{m'_1}} \sum_{r_1 r_2 r_3 = \delta_1} \sum_{e_1 | r_2 r_3} \sum_{e_2 | r_3} \sum_{\substack{g_0 | e_1 e_2 \delta_1 a m'_1 \\ g_0 \equiv 0 \pmod{d}}} \mathcal{T}_c^\pm,$$

and that $\delta_2 = \frac{h a m'_1}{g_0}$, and $h = e_1 e_2 \delta_1 = e_1 e_2 r_1 r_2 r_3$.

Next we analyze the sum over g_0 in all four terms. For the $\text{flrt}(\delta_2)$ = $\text{flrt}(\frac{e_1 e_2 \delta_1 a m'_1}{g_0}) \leq \text{flrt}(e_1 e_2 \delta_1 a m'_1) = \text{flrt}(h a m'_1)$, and otherwise we see that the overall power of g_0 is negative in all terms, and so the smallest value of g_0 , namely d , leads to the dominant part.

Putting this together, and simplifying, we obtain

$$\mathcal{S}_c^\pm \ll q^\varepsilon \sum_a \frac{1}{a^{3/2}} \sum_{c_2} \frac{1}{c_2^{3/2}} \sum_{d|c_2} d \sum_{k_1} k_1^{3/2} \sum_{m'_1} \frac{1}{\sqrt{m'_1}} \sum_{r_1 r_2 r_3 = \delta_1} \sum_{e_1 | r_2 r_3} \sum_{e_2 | r_3} \frac{(M_2 N)^{1/2}}{CK} \left(\frac{\sqrt{aMN}}{C} \right)^\delta \frac{\text{flrt}(ham'_1)}{\sqrt{ham'_1}} \left[(**) + \frac{P^* C}{qd^{3/2} c_2 k_1^3 \sqrt{k_1 k_1^*}} + \frac{P_1^* (P_2^* P_3^*)^{1/2} C}{qk_1^2 c_2 \sqrt{dh_2 h_3 k_1 k_1^*}} + \frac{(P_1^*)^{1/2} P_2^* P_3^* C^{1/2}}{dk_1^{5/2} \sqrt{qh_1 c_2}} \right].$$

Our next goal is to estimate the sum over m'_1 . Since m'_1 is independent of δ_1 (and hence e_1, e_2), we may move the sum over m'_1 to the inside. We shall use the following estimate:

$$\sum_{n \leq X} \frac{\text{flrt}(nN)}{n} \ll \text{flrt}(N)(XN)^\varepsilon,$$

which can be proved by elementary methods. Applying this to the sums over m'_1 , we obtain with easy simplifications

$$(11.23) \quad \mathcal{S}_c^\pm \ll q^\varepsilon \sum_a \frac{1}{a^{3/2}} \sum_{c_2} \frac{1}{c_2^{3/2}} \sum_{d|c_2} d \sum_{k_1} k_1^{3/2} \sum_{r_1 r_2 r_3 = \delta_1} \sum_{e_1 | r_2 r_3} \sum_{e_2 | r_3} \frac{(M_2 N)^{1/2}}{CK} \left(\frac{\sqrt{aMN}}{C} \right)^\delta \frac{\text{flrt}(ha)}{\sqrt{ha}} \left[(**) + \frac{P^* C}{qd^{3/2} c_2 k_1^3 \sqrt{k_1 k_1^*}} + \frac{P_1^* (P_2^* P_3^*)^{1/2} C}{qk_1^2 c_2 \sqrt{dh_2 h_3 k_1 k_1^*}} + \frac{(P_1^*)^{1/2} P_2^* P_3^* C^{1/2}}{dk_1^{5/2} \sqrt{qh_1 c_2}} \right].$$

Using $\text{flrt}(ha) \leq \sqrt{ha}$, we can easily see that the outer variables sum to give no significant contribution. Therefore, we have

$$\mathcal{S}_c^\pm \ll q^\varepsilon \max_a \frac{(M_2 N)^{1/2}}{CK} \left(\frac{\sqrt{aMN}}{C} \right)^\delta \left[(**) + q^{-1} P^* C + q^{-1} P_1^* (P_2^* P_3^*)^{1/2} C + q^{-1/2} (P_1^*)^{1/2} P_2^* P_3^* C^{1/2} \right].$$

Substituting for P_i^* and simplifying, we obtain

$$(11.24) \quad \mathcal{S}_c^\pm \ll q^\varepsilon \max_a \frac{(M_2 N)^{1/2}}{CK} \left(\frac{\sqrt{aMN}}{C} \right)^\delta \left[(**) + \frac{K^3 C}{Nq} + \frac{K^2 C}{qN_1 \sqrt{N_2 N_3}} + \frac{K^{5/2} C^{1/2}}{q^{1/2} N_1^{1/2} N_2 N_3} \right].$$

Now we split once more into the two types of non-oscillatory behavior. The **post-transition** case from Lemma 8.3 has $\frac{\sqrt{aMN}}{C} \gg q^\varepsilon$, which leads to $\delta = -1$ and $K \asymp \frac{(aMN)^{1/2}}{M_2}$. Therefore,

$$\mathcal{S}_c^\pm \ll q^\varepsilon \max_a (M_2 N)^{1/2} \left[(**) + \frac{aMN}{M_2^2 Nq} + \frac{\sqrt{aMN}}{qM_2 N_1 \sqrt{N_2 N_3}} + \frac{\sqrt{aMN}}{q^{1/2} M_2^{3/2} N_1^{1/2} N_2 N_3} \right].$$

This simplifies to give

$$(11.25) \quad \mathcal{S}_c^\pm \ll q^\varepsilon \left[1 + \frac{\sqrt{M_1}}{\sqrt{M_2}} \frac{\sqrt{M_1 a^2 N}}{q} + \frac{M_1^{1/2} \sqrt{a N_2 N_3}}{q} + \frac{\sqrt{M_1}}{\sqrt{M_2}} \frac{\sqrt{a N_1}}{\sqrt{q}} \right].$$

Using $M_1 \ll M_2$, $M_1 \ll q^{1/2+\varepsilon}$, and $aN_i \ll q^{1/2+\varepsilon}$, we deduce that $\mathcal{S}_c^\pm \ll q^\varepsilon$. One may observe that the part of \mathcal{S}_c^\pm arising from $Z_1^* Z_2^0$ and $Z_1^0 Z_2^*$ contributes at most $O(q^{-1/4+\varepsilon})$. With some additional work, one could show the contribution from $Z_1^* Z_2^*$ is also at most $O(q^{-1/4+\varepsilon})$.

For the **non-oscillatory, pre-transition** case from Lemma 8.4 with $\delta = \kappa - 1 \geq 2$ (here is the only place where the choice of $\kappa = 2$ does not work), we have $K \ll q^\varepsilon \frac{C}{M_2}$, and $C \gg q^\varepsilon \sqrt{aMN}$, so we obtain

$$(11.26) \quad \mathcal{S}_c^\pm \ll q^\varepsilon \max_a (M_2 N)^{1/2} \left[(**) + \frac{aMN}{M_2^2 N q} + \frac{\sqrt{aMN}}{q M_2 N_1 \sqrt{N_2 N_3}} + \frac{\sqrt{aMN}}{M_2^{3/2} q^{1/2} N_1^{1/2} N_2 N_3} \right].$$

This is precisely the same bound as in (11.25), and so $\mathcal{S}_c^\pm \ll q^\varepsilon$. Actually, we only need $\kappa - 1 \geq 2$ for the term arising from $Z_1^0 Z_2^0$.

Finally, we consider the **Oscillatory** case (where recall $\delta = \kappa - 1$ and only the $+$ sign enters). For this, we return to (11.9), that is,

$$\begin{aligned} \mathcal{T}_c^+ &\ll \frac{g_0 k_1 c_2}{KC} \left(\frac{\sqrt{aMN}}{C} \right)^{\kappa-1} (M_2 N)^{1/2} \frac{\sqrt{\delta_2}}{h} \sum_{\delta_3 | \frac{c_2}{(g_0, c_2)}} \sum_{(\delta_4, \delta_2 g_0 m'_1)=1} \sum_{\substack{(f, \delta_2 \delta_4 c'_0)=1 \\ (9.1) \text{ is true}}} \int_{|t| \ll Y' q^\varepsilon} \\ &\quad \left(\frac{CK}{a M_1 N} \right)^2 \int_{(1+\varepsilon)} |\widetilde{w}_T(\mathbf{u}, t)| \left| P_1^{u_1} \left(\frac{P_2}{f} \right)^{u_2} \left(\frac{P_3}{f} \right)^{u_3} \left(\frac{C}{\delta_4 c_2} \right)^{u_4} \right| \sum_{\mathbf{c}} |Z_{\mathbf{c}, t}(\mathbf{u})| d\mathbf{u} dt, \end{aligned}$$

where again we should technically move the contours before applying the absolute values. Then we obtain

$$\begin{aligned} \mathcal{T}_c^+ &\ll q^\varepsilon \frac{g_0 k_1 c_2}{KC} \left(\frac{\sqrt{aMN}}{C} \right)^{\kappa-1} (M_2 N)^{1/2} \frac{\sqrt{\delta_2}}{h} \sum_{\delta_3 | \frac{c_2}{(g_0, c_2)}} \sum_{(\delta_4, \delta_2 g_0 m'_1)=1} \left(\frac{CK}{a M_1 N} \right)^2 \\ &\quad \text{flrt}(\delta_2) \text{flrt}(\delta_3) \left[\frac{(\delta_4, q)^{1/2}}{q^{1/2}} Y'^2 \frac{(P_1 P_2 P_3 C)^{1/2} \text{flrt}(\delta_3)^{1/2}}{\sqrt{\delta_2 \delta_4 \delta_5 c_2}} + Y' \frac{(\delta_4, q)}{q} \frac{P_1 P_2 P_3 C}{\delta_2 \delta_4 \delta_5 c_2 \sqrt{k_1 k_1^*}} \right. \\ &\quad \left. + Y' \frac{P_1 (P_2 P_3)^{1/2} C}{\delta_2 \delta_4 c_2 \sqrt{\delta_5}} \frac{(\delta_4, q)}{q \sqrt{k_1 k_1^*}} + Y' \frac{P_1^{1/2} P_2 P_3 C^{1/2}}{\delta_4^{1/2} c_2^{1/2} \delta_2 \delta_5} \frac{(\delta_4, q)^{1/2}}{q^{1/2}} \right]. \end{aligned}$$

Luckily, we may re-use some of the previous analysis in the non-oscillatory cases. We wish to sum over all the outer variables. Note that Y' is independent of them, and we have $P_i \asymp \frac{NaM_i}{CK} \frac{k'_i}{N'_i}$; previously we had $P_i \ll \frac{k'_i}{N'_i}$, so the only difference here is the extra factor $\frac{NaM_i}{CK}$ (which happens to be Y'^2). Therefore, the previous method of bounding the outer variables works identically as in this case. This time the term arising from $Z_1^* Z_2^*$ is identical to (11.15), save for the $\text{flrt}(c_2)$. We again denote it by $(**)$. Therefore, we have by altering (11.24) with the appropriate factors of Y' that

$$\mathcal{S}_c^\pm \ll q^\varepsilon \max_a \frac{(M_2 N)^{1/2}}{CK Y'^4} \left(\frac{\sqrt{aMN}}{C} \right)^{\kappa-1} \left[(**) + \frac{Y'^7 K^3 C}{N q} + \frac{Y'^5 K^2 C}{q N_1 \sqrt{N_2 N_3}} + \frac{Y'^6 K^{5/2} C^{1/2}}{q^{1/2} N_1^{1/2} N_2 N_3} \right].$$

Substituting for Y' , simplifying, and using $K \ll q^\varepsilon \frac{C}{M_2}$, this becomes

$$\begin{aligned} \mathcal{S}_c^\pm &\ll q^\varepsilon \max_a (M_2 N)^{1/2} \left(\frac{\sqrt{aMN}}{C} \right)^{\kappa-1} \\ &\quad \left[(**) + \frac{(M_1 a N)^{3/2}}{C M_2^{1/2} N q} + \frac{(M_1 a N)^{1/2}}{q M_2^{1/2} N_1 \sqrt{N_2 N_3}} + \frac{M_1 a N}{q^{1/2} C M_2^{1/2} N_1^{1/2} N_2 N_3} \right]. \end{aligned}$$

Using $C \gg q^{-\varepsilon} \sqrt{MaN}$ and $\kappa - 1 \geq 1$, this gives

$$\mathcal{S}_c^\pm \ll q^\varepsilon \max_a \left[1 + \frac{M_1 (M_2 a^2 N)^{1/2}}{M_2 q} + \frac{(M_1 a N_2 N_3)^{1/2}}{q} + \frac{\sqrt{M_1}}{\sqrt{M_2}} \frac{\sqrt{aN_1}}{\sqrt{q}} \right].$$

Again as in the non-oscillatory case, we see that $\mathcal{S}_c^\pm \ll q^\varepsilon$.

12. BOUNDING THE DIRICHLET SERIES

12.1. Discrete spectrum. In this section we prove Lemma 11.1. Towards this, we develop some properties of an auxilliary Dirichlet series with the following

Lemma 12.1. *Suppose $N = LM$, f^* is a newform of level M , and d, Q are nonzero integers. For $\ell | L$, let $f = f^*|_\ell$, write $d/\ell = d_1/\ell_1$ in lowest terms (so $d = (d, \ell)d_1$, $\ell = (d, \ell)\ell_1$), and let $d_1 = d_M d_0$ with $d_M | M^\infty$ and $(d_0, M) = 1$. Define the Dirichlet series*

$$(12.1) \quad Z_{d,\ell,Q}(s, u) := \sum_{\substack{m,n \geq 1 \\ (n,Q)=1}} \frac{\nu_{f^*|_\ell}(dmn)}{m^s n^u},$$

initially for $\operatorname{Re}(s), \operatorname{Re}(u)$ large. Then $Z_{d,\ell,Q}$ has analytic continuation to $\operatorname{Re}(s), \operatorname{Re}(u) \geq \sigma > 1/2$, wherein it satisfies the bound

$$(12.2) \quad |Z_{d,\ell,Q}(s, u)| \ll_\sigma |\nu_{f^*}(1)| (d, \ell)^{1/2} d_M^{-1/2} d_0^\theta (dNQ)^\varepsilon |L(f^*, s) L(f^*, u)|.$$

Proof. Firstly, from (4.45), we have

$$Z_{d,\ell,Q}(s, u) = \sum_{\substack{m,n \geq 1 \\ (n,Q)=1}} \frac{\ell^{1/2} \nu_{f^*}(dmn/\ell)}{m^s n^u}.$$

We have $\nu_{f^*}(dmn/\ell) = \nu_{f^*}(d_1 mn/\ell_1) = \lambda_{f^*}(d_M) \nu_{f^*}(d_0 mn/\ell_1)$, and so

$$Z_{d,\ell,Q}(s, u) = \ell^{1/2} \lambda_{f^*}(d_M) \nu_{f^*}(1) \sum_{\substack{m,n \geq 1 \\ (n,Q)=1}} \frac{\lambda_{f^*}(d_0 mn/\ell_1)}{m^s n^u}.$$

Using (4.49), and complete multiplicativity of Hecke eigenvalues for primes dividing M , we get that $|\lambda_{f^*}(d_M)| \leq d_M^{-1/2}$.

By an exercise with the Hecke relations (somewhat in the spirit of (3.2)), one may derive the analytic continuation and the bound

$$\sum_{\substack{m,n \geq 1 \\ (n,Q)=1}} \frac{\lambda_{f^*}(d_0 mn/\ell_1)}{m^s n^u} \ll_\sigma \frac{(dNQ)^\varepsilon}{\ell_1^{1/2}} d_0^\theta |L(f^*, s) L(f^*, u)|,$$

where recall that $\operatorname{Re}(s), \operatorname{Re}(u) \geq \sigma > 1/2$. □

Now we proceed to prove Lemma 11.1. Recall the definition

$$(12.3) \quad Z_j(\mathbf{u}) = \left(\sum_{\substack{(c'_0, f g_0 m'_1)=1 \\ \delta_4 c'_0 \equiv 0 \pmod{q k_1 k_1^*}}} \sum_{p_1 \geq 1} \frac{\nu_{\infty,j}(p_1 \delta_4 c'_0)}{p_1^{u_1} c_0^{u_4}} \right) \left(\sum_{p_2, p_3 \geq 0} \frac{\nu_{\frac{1}{\delta_5},j}(p_2 p_3)}{p_2^{u_2} p_3^{u_3}} \right).$$

We begin by decomposing into newforms. By the choice of basis from Section 4.7, we have

$$(12.4) \quad \sum_{|t_j| \leq T} |Z_j(\mathbf{u})| \ll \sum_{AB=\delta_2\delta_5} \sum_{f^* \text{ new, level } B} \sum_{|t_{f^*}| \leq T} \times \sum_{\substack{\ell|A \\ \ell'|A}} \left| \left(\sum_{\substack{(c'_0, fg_0m'_1)=1 \\ \delta_4 c'_0 \equiv 0 \pmod{qk_1k_1^*}}} \sum_{p_1 \geq 1} \frac{\nu_{\infty, f^*|_{\ell}}(p_1 \delta_4 c'_0)}{p_1^{u_1} c_0^{u_4}} \right) \left(\sum_{p_2, p_3 \geq 0} \frac{\nu_{\frac{1}{\delta_5}, f^*|_{\ell'}}(p_2 p_3)}{p_2^{u_2} p_3^{u_3}} \right) \right|.$$

By Lemmas 4.9 and 12.1, we see that the sum over p_2, p_3 has analytic continuation to the desired region, and satisfies

$$\sum_{p_2, p_3 \geq 1} \frac{\nu_{\frac{1}{\delta_5}, f^*|_{\ell'}}(p_2 p_3)}{p_2^{\alpha} p_3^{\beta}} \ll |\nu_{f^*}(1)| |L(f^*, \alpha) L(f^*, \beta)|,$$

uniformly in ℓ' and δ_5 .

The first product in (12.4) is a bit trickier. Recall from (6.13) that $(q, k_1) = 1$ and from (7.9) that $k_1 | \delta_2$. We apply (12.2), with $d = D \frac{\delta_4}{(D, \delta_4)}$, and $D = qk_1k_1^*$. This gives

$$\sum_{\substack{(c'_0, fg_0m'_1)=1 \\ \delta_4 c'_0 \equiv 0 \pmod{qk_1k_1^*}}} \sum_{p_1 \geq 1} \frac{\nu_{f^*|_{\ell}}(p_1 \delta_4 c'_0)}{p_1^{u_1} c_0^{u_4}} \ll q^{\varepsilon} |\nu_{f^*}(1)| \left(\frac{(D, \delta_4)}{D} \right)^{1/2} \frac{(d, \ell)^{1/2} d_0^{\theta}}{d_B^{1/2}} |L(f^*, u_1) L(f^*, u_4)|,$$

where $\frac{d}{\ell} = \frac{d_1}{\ell_1}$ is in lowest terms, and then we factor $d_1 = d_B d_0$ where $d_B | B^{\infty}$, and $(d_0, B) = 1$.

Using $|\nu_{f^*}(1)|^2 = (AB)^{-1} q^{o(1)}$ via (4.42), we then have

$$\begin{aligned} \sum_{\substack{t_j \text{ level } \delta_2\delta_5 \\ |t_j| \leq T}} |Z_j(\mathbf{u})| &\ll q^{\varepsilon} \left(\frac{(D, \delta_4)}{D} \right)^{1/2} \sum_{AB=\delta_2\delta_5} \sum_{\ell|A} (d, \ell)^{1/2} d_B^{-1/2} d_0^{\theta} \\ &\times \frac{1}{AB} \sum_{\substack{f^* \text{ new, level } B \\ |t_{f^*}| \leq T}} |L(f^*, u_1) L(f^*, u_2) L(f^*, u_3) L(f^*, u_4)|. \end{aligned}$$

A standard argument with the spectral large sieve (e.g., see [29, Theorem 3.4] for the level 1 case) implies

$$(12.5) \quad \sum_{\substack{t_j \text{ level } \delta_2\delta_5 \\ |t_j| \leq T}} |Z_j(\mathbf{u})| \ll T^{2+\varepsilon} q^{\varepsilon} \left(\frac{(D, \delta_4)}{D} \right)^{1/2} \sum_{AB=\delta_2\delta_5} \frac{1}{A} \sum_{\ell|A} (d, \ell)^{1/2} d_B^{-1/2} d_0^{\theta}.$$

At this point, the proof of Lemma 12.1 has reduced to elementary estimates with arithmetic functions. The factorization $d = (d, \ell) d_B d_0$ is dependent on ℓ , so it takes some work to estimate the sum over ℓ . To this end, we also factor d in an alternative way, independently of ℓ , by $d = d' f g h$ where $(d', AB) = 1$, $f | A^{\infty}$, $(f, B) = 1$, $g | B^{\infty}$, $(g, A) = 1$, and $h | A^{\infty}$ and $h | B^{\infty}$. Note that $(f, g) = (f, h) = (g, h) = 1$. Then writing the old variables in terms of these, we have

$$d = \underbrace{(f, \ell)(h, \ell)}_{(d, \ell)} \underbrace{g \frac{h}{(h, \ell)}}_{d_B} \underbrace{d' \frac{f}{(f, \ell)}}_{d_0}.$$

Inserting this into (12.5), and summing over $\ell|L$, we obtain

$$\sum_{\substack{t_j \text{ level } \delta_2 \delta_5 \\ |t_j| \leq T}} |Z_j(\mathbf{u})| \ll T^{2+\varepsilon} q^\varepsilon d^\theta \left(\frac{(D, \delta_4)}{D} \right)^{1/2} \sum_{AB=\delta_2 \delta_5} \frac{1}{A} \frac{(f, A)^{1/2-\theta} f^\theta(h, A)}{\sqrt{gh}}.$$

Writing $d'f = \frac{D}{(D, \delta_4)} \delta_4 \frac{1}{gh}$, we obtain

$$\sum_{\substack{t_j \text{ level } \delta_2 \delta_5 \\ |t_j| \leq T}} |Z_j(\mathbf{u})| \ll T^{2+\varepsilon} q^\varepsilon \left(\frac{(D, \delta_4)}{D} \right)^{1/2-\theta} \delta_4^\theta \sum_{AB=\delta_2 \delta_5} \frac{1}{A} \frac{(f, A)^{1/2-\theta} (h, A)}{(gh)^{1/2+\theta}}.$$

Now let us pause to gauge our progress towards (11.10). The inner sum over A easily gives $O(q^\varepsilon)$, and so if we trivially bound this part, and use $D = qk_1k_1^*$ (also recall $(k_1, \delta_4) = 1$ from (9.3)), we get the bound

$$\sum_{t_j} |Z_j(\mathbf{u})| \ll q^{\theta-\frac{1}{2}} \frac{1}{(k_1k_1^*)^{1/2}} (q, \delta_4)^{1/2-\theta} (k_1k_1^*)^\theta \delta_4^\theta T^2 q^\varepsilon.$$

so it is required to save $\delta_4^{\frac{1}{2}+\theta} (k_1k_1^*)^\theta$, which must come from better-estimating the sum over A .

This inner sum over A, B may be factored into prime powers. For the primes $p \nmid k_1\delta_4$, all we use is that the local factor is ≤ 1 (leading to a $O(q^\varepsilon)$ bound from these primes, by the observation in the previous paragraph). Recall $\delta_4|\delta_2\delta_5$ since $\delta_5 = [\delta_3, \delta_4]$ (see (9.2)), and $\delta_4|d$, so $\delta_4|fgh$. For $p|\delta_4$, say $p^\nu||\delta_4$, $p^f||f$, and so on for g, h, A , and B , by an abuse of notation. Now the variables in the exponents are written additively. Since $\delta_4|fgh$, in additive notation we have $f + g + h \geq \nu$. Also, $A + B \geq \nu$, since $\delta_4|\delta_2\delta_5$.

In the case $B = 0$, then $g = h = 0$ and the local factor is

$$\frac{1}{p^A} (p^f, p^A)^{\frac{1}{2}-\theta} \leq p^{\nu(-\frac{1}{2}-\theta)},$$

which is the local factor of $\delta_4^{-\frac{1}{2}-\theta}$. In the case $A = 0$, the local factor is no larger than the local factor of $\delta_4^{-\frac{1}{2}-\theta}$ as can be seen easily. Finally, if $A, B > 0$, then $f = g = 0$, $h \geq \nu$, and so the local factor equals

$$\frac{1}{p^A} \min(p^h, p^A) p^{-h(\frac{1}{2}+\theta)} \leq p^{-\nu(\frac{1}{2}+\theta)}.$$

Now suppose that $p|k_1$. Then since $k_1|\delta_2$ and $k_1|d$ (whence $k_1|fgh$) essentially the same proof used for δ_4 shows the local factors for primes dividing k_1 give $O(k_1^{-1/2})$.

In summary, this shows

$$\sum_{\substack{t_j \text{ level } \delta_2 \delta_5 \\ |t_j| \leq T}} |Z_j(\mathbf{u})| \ll T^2 q^\varepsilon \left(\frac{(q, \delta_4)}{qk_1k_1^*} \right)^{1/2-\theta} (k_1\delta_4)^{-1/2}.$$

Using $(k_1k_1^*)^\theta \leq k_1^{2\theta} \leq k_1^{1/2}$ gives (11.10), as desired.

12.2. Continuous spectrum. Here we prove Lemma 11.2. By (4.26) and (4.28), we have

$$\nu_{\text{ab}}(n, u) = \alpha(u) \phi_{\text{ab}}(n, u) |n|^{u-\frac{1}{2}}, \quad \text{where} \quad \alpha(u) = 2\pi^u \frac{(\Gamma(u)\Gamma(1-u))^{1/2}}{\Gamma(u)}.$$

Note that $|\alpha(1/2 + it)| = 2\sqrt{\pi}$ is independent of $t \in \mathbb{R}$. Define

$$(12.6) \quad Z_1 = Z_1(\alpha, \beta) = \sum_{m, n \geq 1} \frac{\nu_{1/r, \mathfrak{c}}(mn, 1/2 + it)}{m^\alpha n^\beta},$$

and

$$(12.7) \quad Z_2 = Z_2(\alpha, \beta) = \sum_{\substack{m, n \geq 1 \\ \delta m \equiv 0 \pmod{D} \\ (m, Q) = 1}} \frac{\nu_{\infty, \mathfrak{c}}(\delta mn, 1/2 + it)}{m^\alpha n^\beta},$$

where we assume the level is N as in Section 4.8. This meaning of N is valid only within the confines of this subsection, and hence should not be confused with the dyadic variable N in the rest of the article. For Lemma 11.2, we shall need $N = \delta_2 \delta_5$, $Q = fg_0 m'_1$, $D = qk_1 k_1^*$, $\delta = \delta_4$, and $r = \delta_5$, but we do need make these specifications yet. With this notation, we have $Z_{\mathfrak{c}, t}(\mathbf{u}) = Z_1(u_2, u_3) Z_2(u_1, u_4)$.

The plan of the proof is to first derive bounds on $Z_1^0, Z_1^*, Z_2^0, Z_2^*$ individually, and follow this with estimates for the sums over \mathfrak{c} . Using (4.65), we have (with $u = \frac{1}{2} + it$)

$$(12.8) \quad Z_1 = \frac{f'_r}{(f'_r, r')} \frac{s'_f}{(s'_f, f_s)} \frac{\alpha(u)}{(N'' s f_r^2)^u} \sum_{k_r | (f'_r, r')} \sum_{k_s | (s'_f, f_s)} \frac{1}{\varphi(\frac{(f'_r, r')}{k_r})} \frac{1}{\varphi(\frac{(s'_f, f_s)}{k_s})} \sum_{\chi \pmod{\frac{(f'_r, r')}{k_r}}} \sum_{\psi \pmod{\frac{(s'_f, f_s)}{k_s}}} \frac{\tau(\overline{\chi}) \tau(\overline{\psi})}{L(2u, \overline{\chi^2 \psi^2} \chi_0)} (\chi \psi) (\overline{s_0 f_0 w'}) \chi(-k_s \overline{(s'_f, f_s)}) \psi(k_r \overline{(f'_r, r')}) \sum_{(d, f_s r') = 1} (\overline{\chi \psi}) (d^2) d^{1-2u} \sum_{\substack{m, n \geq 1 \\ (*)}} \frac{(\chi \psi)(\ell) S(\ell, 0; s_0 f_0)}{m^\alpha n^\beta} (mn)^{u-\frac{1}{2}}.$$

Here $(*)$ stands for the following conditions: We have $mn = \frac{f'_r}{(f'_r, r')} \frac{s'_f}{(s'_f, f_s)} k_r k_s \ell$, and also $\ell \equiv 0 \pmod{d}$. Write $Z_1 = Z_1^0 + Z_1^*$ where Z_1^0 corresponds to the part with both χ and ψ principal.

By a trivial bound, we have for $\text{Re}(\alpha), \text{Re}(\beta) \geq \sigma > 1$,

$$(12.9) \quad |Z_1^0| \ll_\sigma \frac{N^\varepsilon}{f_r \sqrt{s N''} (f'_r, r') (s'_f, f_s)} \frac{1}{|\zeta(1 + 2it)|} \ll_\sigma \frac{(N(1 + |t|))^\varepsilon}{(f, \frac{N}{f}) f_r \sqrt{s N''}}.$$

Meanwhile, we have

$$(12.10) \quad |Z_1^*| \ll \frac{f'_r}{(f'_r, r')} \frac{s'_f}{(s'_f, f_s)} \frac{N^\varepsilon}{f_r \sqrt{s N''}} \sum_{k_r | (f'_r, r')} \sum_{k_s | (s'_f, f_s)} \frac{1}{\varphi(\frac{(f'_r, r')}{k_r})} \frac{1}{\varphi(\frac{(s'_f, f_s)}{k_s})} \sum'_{\chi \pmod{\frac{(f'_r, r')}{k_r}}} \sum'_{\psi \pmod{\frac{(s'_f, f_s)}{k_s}}} \frac{|\tau(\overline{\chi}) \tau(\overline{\psi})|}{|L(1 + 2it, \overline{\chi^2 \psi^2} \chi_0)|} |Y_1|,$$

where the notation \sum' means the principal character is omitted, and with

$$(12.11) \quad V = \frac{f'_r}{(f'_r, r')} \frac{s'_f}{(s'_f, f_s)} k_r k_s,$$

we have

$$(12.12) \quad Y_1 = \sum_{(d, f_s r')=1} (\overline{\chi\psi})(d^2) d^{1-2u} \sum_{mn \equiv 0 \pmod{dV}} \frac{(\chi\psi)(\frac{mn}{V}) S(\frac{mn}{V}, 0; s_0 f_0)}{m^\alpha n^\beta} (mn)^{u-\frac{1}{2}}.$$

Moreover, the analytic continuation of Z_1^* will be inherited from that of Y_1 .

Similarly to (7.4) and (7.5), one can show the formal identity

$$(12.13) \quad \sum_{mn \equiv 0 \pmod{D}} f(m, n) = \sum_{CAB=D} \mu(C) \sum_{m, n} f(CAm, CBn).$$

Applying this to Y_1 with $D = dV$, we obtain

$$(12.14) \quad Y_1 = V^{u-\frac{1}{2}} \sum_{(d, f_s r')=1} (\overline{\chi\psi})(d) d^{\frac{1}{2}-u} \sum_{CAB=dV} \frac{\mu(C)(\chi\psi)(C) C^{u-\frac{1}{2}}}{A^\alpha B^\beta C^{\alpha+\beta}} \sum_{m, n} \frac{(\chi\psi)(mn) S(Cdmn, 0; s_0 f_0)}{m^\alpha n^\beta (mn)^{\frac{1}{2}-u}}.$$

One can readily observe that $(d, V) = 1$, and $(V, s_0 f_0) = 1$. In the factorization $CAB = dV$, one may split each of C, A, B uniquely into its part dividing d and dividing V separately, and thereby factor the sum. In this way, we obtain (with $\text{Re}(u) = \frac{1}{2}$)

$$(12.15) \quad |Y_1| \ll \frac{N^\varepsilon}{V^\sigma} \left| \sum_{(CAB, f_s r')=1} \frac{\mu(C)(\overline{\chi\psi})(AB)}{A^\alpha B^\beta C^{\alpha+\beta} (AB)^{u-\frac{1}{2}}} \sum_{m, n} \frac{(\chi\psi)(mn) S(C^2 ABmn, 0; s_0 f_0)}{m^\alpha n^\beta (mn)^{\frac{1}{2}-u}} \right|.$$

Next we open the Ramanujan sum as a divisor sum, giving

$$(12.16) \quad |Y_1| \ll \frac{N^\varepsilon}{V^\sigma} \sum_{g|s_0 f_0} g \left| \sum_{(CAB, f_s r')=1} \sum_{\substack{m, n \\ C^2 ABmn \equiv 0 \pmod{g}}} \frac{\mu(C)(\overline{\chi\psi})(AB)}{A^\alpha B^\beta C^{\alpha+\beta} (AB)^{u-\frac{1}{2}}} \frac{(\chi\psi)(mn)}{m^\alpha n^\beta (mn)^{\frac{1}{2}-u}} \right|.$$

Now it is not difficult to see the analytic continuation of Y_1 to $\text{Re}(\alpha, \beta) \geq \sigma = 1/2 + \varepsilon$, and therein we obtain the bound

$$(12.17) \quad |Y_1| \ll N^\varepsilon \left(\frac{(f'_r, r')}{f'_r} \frac{(s'_f, f_s)}{s'_f} \right)^{1/2} \frac{(s_0 f_0)^{1/2}}{(k_r k_s)^{1/2}} |L(\alpha - it, \chi\psi) L(\alpha + it, \overline{\chi\psi}) L(\beta - it, \chi\psi) L(\beta + it, \overline{\chi\psi})|.$$

We recall the well-known bound on the fourth moment of Dirichlet L -functions, namely

$$(12.18) \quad \sum_{\chi \pmod{N}} |L(\sigma + it, \chi)|^4 \ll_{\sigma, \varepsilon} (1 + |t|)^{1+\varepsilon} N^{1+\varepsilon},$$

for $\sigma \geq 1/2$. Moreover, we have the hybrid version

$$(12.19) \quad \int_{|t| \leq T} \sum_{\chi \pmod{N}} |L(\sigma + it, \chi)|^4 \ll_{\sigma, \varepsilon} (1 + T)^{1+\varepsilon} N^{1+\varepsilon}.$$

For references, consult [27, Chapter 10] or [12, Theorem 2]; the statements in these sources do not precisely claim these bounds, but the methods can be easily modified. In addition,

we have

$$(12.20) \quad \frac{1}{|L(1+2it, \chi)|} \ll (1+|t|)^\varepsilon N^\varepsilon,$$

for which see [28, Theorem 11.4]. Thus applying Hölder's inequality, and the bound (12.18) we obtain

$$(12.21) \quad |Z_1^*| \ll \frac{N^\varepsilon(1+|t|)^{1+\varepsilon}}{f_r \sqrt{sN''}} (f_r s')^{1/2} = \frac{N^\varepsilon(1+|t|)^{1+\varepsilon}}{\sqrt{fN''}}.$$

Using (12.19), we alternatively have

$$(12.22) \quad \int_{|t| \leq T} |Z_1^*| \ll \frac{q^\varepsilon T^{1+\varepsilon}}{\sqrt{fN''}}.$$

Recalling (4.55), we have

$$(12.23) \quad fN'' = (w, N)N'' = \frac{N}{(w, \frac{N}{(w, N)})} = \frac{N}{(f, \frac{N}{f})}.$$

Next we study Z_2 , which is more difficult than Z_1 . We have $\mathfrak{a} = \infty \sim 1/N$, so $r = N$ and $s = 1$, and also $f_r = f$, $r' = N'$. First we perform a minor simplification by writing the congruence $\delta m \equiv 0 \pmod{D}$ as $m \equiv 0 \pmod{\frac{D}{(\delta, D)}}$ (so necessarily $(Q, \frac{D}{(\delta, D)}) = 1$ otherwise the sum is empty). Then we have

$$(12.24) \quad Z_2 = \left(\frac{(D, \delta)}{D}\right)^\alpha Y_2, \quad \text{where} \quad Y_2 := \sum_{\substack{m, n \geq 1 \\ (m, Q) = 1}} \frac{\nu_{\infty, \mathfrak{c}}(amn, \frac{1}{2} + it)}{m^\alpha n^\beta},$$

and where

$$(12.25) \quad a = \frac{\delta D}{(\delta, D)} = [\delta, D].$$

Applying (4.65), we obtain

$$(12.26) \quad Y_2 = \frac{f'_N}{(f'_N, N')} \frac{\alpha(u)}{(N'' f^2)^u} \sum_{k_N | (f'_N, N')} \frac{1}{\varphi(\frac{(f'_N, N')}{k_N})} \sum_{\chi \pmod{\frac{(f'_N, N')}{k_N}}} \frac{\tau(\overline{\chi}) \chi(-\overline{f_0} w')}{L(2u, \overline{\chi^2} \chi_0)} \\ \sum_{(d, N')=1} d^{1-2u} \overline{\chi}(d^2) \sum_{(*)} \frac{\chi(\ell) S(\ell, 0; f_0)}{m^\alpha n^\beta} (amn)^{u-\frac{1}{2}}.$$

Now the symbol $(*)$ stands for the following conditions. We have $amn = \frac{f'_N}{(f'_N, N')} k_N \ell$, $amn \equiv 0 \pmod{d}$, and also the condition $(m, Q) = 1$. From the condition $(d, N') = 1$, we equivalently obtain that $\ell \equiv 0 \pmod{d}$. Define

$$(12.27) \quad b = \frac{f'_N}{(f'_N, N')} k_N = \frac{f'_N}{\frac{(f'_N, N')}{k_N}},$$

and write

$$(12.28) \quad a = (a, b)a', \quad \text{and} \quad b = (a, b)b',$$

so that the condition $b|amn$ is equivalent to $b'|mn$. Then $\ell = a' \frac{mn}{b'}$, and we have

$$(12.29) \quad Y_2 = \frac{f'_N}{(f'_N, N')} \frac{\alpha(u) a^{u-\frac{1}{2}}}{(N'' f^2)^u} \sum_{k_N | (f'_N, N')} \frac{1}{\varphi\left(\frac{(f'_N, N')}{k_N}\right)} \sum_{\chi \pmod{\frac{(f'_N, N')}{k_N}}} \frac{\tau(\overline{\chi}) \chi(\overline{f_0} w' a')}{L(2u, \chi^2 \chi_0)} X_2,$$

where

$$(12.30) \quad X_2 := \sum_{(d, N')=1} d^{1-2u} \overline{\chi}(d^2) \sum_{\substack{mn \equiv 0 \pmod{b'} \\ amn \equiv 0 \pmod{d}}} \frac{\chi\left(\frac{mn}{b'}\right) S(a' \frac{mn}{b'}, 0; f_0)}{m^\alpha n^\beta} (mn)^{u-\frac{1}{2}}.$$

Here $(b', f_0 d) = 1$, since $b|(N')^\infty$. Opening the Ramanujan sum, we have

$$(12.31) \quad X_2 = \sum_{e|f_0} e\mu(f_0/e) \sum_{(d, N')=1} d^{1-2u} \overline{\chi}(d^2) \sum_{\substack{mn \equiv 0 \pmod{b'} \\ amn \equiv 0 \pmod{d} \\ a'mn \equiv 0 \pmod{e}}} \frac{\chi\left(\frac{mn}{b'}\right)}{m^\alpha n^\beta} (mn)^{u-\frac{1}{2}}.$$

Let $g = (a, d)$, so that

$$(12.32) \quad X_2 = \sum_{\substack{g|a \\ (g, N')=1}} g^{1-2u} \overline{\chi}(g^2) \sum_{e|f_0} e\mu(f_0/e) \sum_{(d, N' \frac{a}{g})=1} d^{1-2u} \overline{\chi}(d^2) \sum_{\substack{mn \equiv 0 \pmod{b'} \\ mn \equiv 0 \pmod{d} \\ a'mn \equiv 0 \pmod{e}}} \frac{\chi\left(\frac{mn}{b'}\right)}{m^\alpha n^\beta} (mn)^{u-\frac{1}{2}}.$$

Applying (12.13) to X_2 with the modulus d , we obtain

$$(12.33) \quad X_2 = \sum_{\substack{g|a \\ (g, N')=1}} g^{1-2u} \overline{\chi}(g^2) \sum_{e|f_0} e\mu(f_0/e) \sum_{(d, N' \frac{a}{g})=1} d^{1-2u} \overline{\chi}(d^2) \sum_{CAB=d} \mu(C) \\ \sum_{\substack{Cdmn \equiv 0 \pmod{b'} \\ a'Cdmn \equiv 0 \pmod{e}}} \frac{\chi\left(\frac{Cdmn}{b'}\right)}{(CAm)^\alpha (CBn)^\beta} (Cdmn)^{u-\frac{1}{2}}.$$

Since $C|d$, $(d, N') = 1$, and $b'|(N')^\infty$, the congruence $Cdmn \equiv 0 \pmod{b'}$ is equivalent to $mn \equiv 0 \pmod{b'}$. We can then write X_2 as

$$(12.34) \quad X_2 = \sum_{\substack{g|a \\ (g, N')=1}} g^{1-2u} \overline{\chi}(g^2) \sum_{e|f_0} e\mu(f_0/e) \sum_{(CAB, N' \frac{a}{g})=1} \frac{\mu(C) \overline{\chi}(AB) (AB)^{\frac{1}{2}-u}}{C^{\alpha+\beta} A^\alpha B^\beta} \\ \sum_{\substack{mn \equiv 0 \pmod{b'} \\ a'C^2 ABmn \equiv 0 \pmod{e}}} \frac{\chi\left(\frac{mn}{b'}\right)}{m^\alpha n^\beta} (mn)^{u-\frac{1}{2}}.$$

Next we use (12.13) again, this time on the congruence modulo b' , giving

$$(12.35) \quad X_2 = (b')^{u-\frac{1}{2}} \sum_{\substack{g|a \\ (g, N')=1}} g^{1-2u} \overline{\chi}(g^2) \sum_{e|f_0} e \mu(f_0/e) \sum_{xyz=b'} \frac{\mu(x) \chi(x) x^{u-\frac{1}{2}}}{x^{\alpha+\beta} y^{\alpha} z^{\beta}} \\ \sum_{(CAB, N' \frac{a}{g})=1} \frac{\mu(C) \overline{\chi}(AB) (AB)^{\frac{1}{2}-u}}{C^{\alpha+\beta} A^{\alpha} B^{\beta}} \sum_{a' C^2 A B x b' m n \equiv 0 \pmod{e}} \frac{\chi(mn)}{m^{\alpha} n^{\beta}} (mn)^{u-\frac{1}{2}}.$$

Similarly to the Z_1 case, one can see the meromorphic continuation with a pole only in the case χ is principal. In addition, we have the bound (with $u = \frac{1}{2} + it$)

$$(12.36) \quad |X_2| \ll_{\sigma} \frac{N^{\varepsilon}}{(b')^{\sigma}} |L(\alpha - it, \chi) L(\alpha + it, \overline{\chi}) L(\beta - it, \chi) L(\beta + it, \overline{\chi})| \sum_{e|f_0} e \left(\frac{(a', e)}{e} \right)^{\sigma}.$$

Note

$$(12.37) \quad \sum_{e|f_0} e \left(\frac{(a', e)}{e} \right)^{\sigma} \ll N^{\varepsilon} (a', f_0)^{\sigma} (1 + f_0)^{1-\sigma}.$$

Now write $Z_2 = Z_2^0 + Z_2^*$ where Z_2^0 corresponds to the principal characters, and similarly write $Y_2 = Y_2^0 + Y_2^*$. For $\text{Re}(\alpha, \beta) \geq \sigma = 1 + \varepsilon$, we have trivially

$$(12.38) \quad X_2 \ll N^{\varepsilon} \frac{(a', f_0)}{b'} = N^{\varepsilon} \frac{(a, f_0 b)}{b},$$

recalling (12.28). Thus

$$(12.39) \quad Y_2^0 \ll \frac{f'_N}{(f'_N, N')} \frac{N^{\varepsilon}}{\sqrt{N''} f^2} \sum_{k_N | (f'_N, N')} \frac{k_N}{(f'_N, N')} \frac{(f'_N, N')(a, f_0 b)}{f'_N k_N}.$$

Here b is a function of k_N (cf. (12.27)), and is maximal when $k_N = (f'_N, N')$ in which case $b = f'_N$. Recalling $f_0 f'_N = f$, which implies $(a, f_0 b) \leq (a, f)$, and using $(f'_N, N') = (f, \frac{N}{f})$, in all we obtain

$$(12.40) \quad Y_2^0 \ll \frac{N^{\varepsilon}(a, f)}{(f, \frac{N}{f}) \sqrt{N''} f^2} \frac{1}{|\zeta(1 + 2it)|}.$$

Finally, we obtain

$$(12.41) \quad Z_2^0 \ll \frac{N^{\varepsilon}(1 + |t|)^{\varepsilon}}{(f, \frac{N}{f}) f \sqrt{N''}} ([\delta, D], f) \frac{(\delta, D)}{D}.$$

Using (12.18) and Hölder's inequality, we have with $\sigma = 1/2 + \varepsilon$

$$(12.42) \quad |Y_2^*| \ll \frac{f'_N}{(f'_N, N')} \frac{N^{\varepsilon}(1 + |t|)^{1+\varepsilon}}{(N'' f^2)^{1/2}} \sum_{k_N | (f'_N, N')} \left(\frac{(f'_N, N')}{k_N} \right)^{1/2} \frac{(a', f_0)^{1/2} f_0^{1/2}}{b^{1/2}}.$$

Note the simplification

$$(12.43) \quad \frac{f'_N}{(f'_N, N')} \left(\frac{(f'_N, N')}{k_N} \right)^{1/2} \frac{f_0^{1/2}}{b^{1/2}} = f^{1/2} \frac{(a, b)^{1/2}}{k_N}.$$

We have $(a', f_0) = (a, f_0)$ since $(b, f_0) = 1$, and $\frac{(a, b)^{1/2}}{k_N} \leq (a, f'_N)^{1/2}$. Thus

$$(12.44) \quad |Y_2^*| \ll \frac{N^\varepsilon(1+|t|)^{1+\varepsilon}}{(N''f)^{1/2}}(a, f)^{1/2}.$$

Hence

$$(12.45) \quad |Z_2^*| \ll \left(\frac{(\delta, D)}{D}\right)^{1/2} \frac{N^\varepsilon(1+|t|)^{1+\varepsilon}}{(N''f)^{1/2}} \left(\frac{\delta D}{(\delta, D)}, f\right)^{1/2}.$$

We also have in a similar way to the Z_1^* case

$$(12.46) \quad \int_{|t| \leq T} |Z_2^*| dt \ll \left(\frac{(\delta, D)}{D}\right)^{1/2} \frac{N^\varepsilon T^{1+\varepsilon}}{(N''f)^{1/2}} \left(\frac{\delta D}{(\delta, D)}, f\right)^{1/2}.$$

Now we proceed to prove the desired bounds in Lemma 11.2. By the work from Section 4.8, the cusps are parameterized by $\frac{1}{uf}$ with $f|N$ and $u \pmod{(f, \frac{N}{f})}$ (with u coprime to N). During the course of the proof, it will be helpful to refer to the following divisor-sum bounds:

$$(12.47) \quad \sum_{d|N} \frac{(d, \frac{N}{d})^2}{N} \ll N^\varepsilon \frac{\text{flrt}(N)}{\sqrt{N}}, \quad \text{and} \quad \sum_{d|N} \frac{(d, \frac{N}{d})^2 d^{1/2}}{N} \ll N^\varepsilon \frac{\text{flrt}(N)^{3/2}}{\sqrt{N}},$$

which in turn can be checked prime-by-prime by multiplicativity. If desired, the former inequality could be bounded by $N^\varepsilon \frac{\text{flrt}(N)^2}{N}$. Along the same lines, we note

$$(12.48) \quad \sum_{d|N} \frac{(d, \frac{N}{d})^2}{d^{1/2}N} \ll N^\varepsilon \frac{\text{flrt}(N)^{3/2}}{N},$$

as well as

$$(12.49) \quad \sum_{d|N} \frac{(d, \frac{N}{d}) d^{1/2}}{N} \ll N^\varepsilon \frac{\sqrt{\text{flrt}(N)}}{\sqrt{N}}, \quad \text{and} \quad \sum_{d|N} \frac{(d, \frac{N}{d})}{N} \ll N^\varepsilon \frac{\text{flrt}(N)}{N}.$$

Combining (12.22) and (12.45), we obtain

$$(12.50) \quad \int_{|t| \leq T} \sum_{\mathfrak{c}} |Z_1^* Z_2^*| dt \ll N^\varepsilon T^{2+\varepsilon} \left(\frac{(\delta, D)}{D}\right)^{1/2} \sum_{f|N} \frac{(f, \frac{N}{f})^2}{N} \left(\frac{\delta D}{(\delta, D)}, f\right)^{1/2}.$$

We have $N = \delta_2 \delta_5$, $D = q k_1 k_1^*$, $\delta = \delta_4 | \delta_5$. Then with these substitutions, and recalling $(\delta_4, k_1) = 1$ we obtain

$$(12.51) \quad \int_{|t| \leq T} \sum_{\mathfrak{c}} |Z_1^* Z_2^*| dt \ll q^\varepsilon T^{2+\varepsilon} \left(\frac{(\delta_4, q)}{q k_1 k_1^*}\right)^{1/2} \sum_{f| \delta_2 \delta_5} \frac{(f, \frac{\delta_2 \delta_5}{f})^2}{\delta_2 \delta_5} \left(k_1 k_1^* \frac{\delta_4 q}{(\delta_4, q)}, f\right)^{1/2}.$$

By multiplicativity, and using $(\delta_2, \delta_5) = 1 = (\delta_2, q)$, and $k_1 | \delta_2$, the inner sum over f factors and simplifies as

$$\sum_{f| \delta_2 \delta_5} \frac{(f, \frac{\delta_2 \delta_5}{f})^2}{\delta_2 \delta_5} \left(k_1 k_1^* \frac{\delta_4 q}{(\delta_4, q)}, f\right)^{1/2} = \left(\sum_{g| \delta_2} \frac{(g, \frac{\delta_2}{g})^2 (g, k_1 k_1^*)^{1/2}}{\delta_2}\right) \left(\sum_{h| \delta_5} \frac{(h, \frac{\delta_5}{h})^2}{\delta_5} \left(\frac{\delta_4 q}{(\delta_4, q)}, h\right)^{1/2}\right).$$

Using $(g, k_1 k_1^*)^{1/2} \leq \sqrt{k_1 k_1^*}$, $(\frac{\delta_4 q}{(\delta_4, q)}, h)^{1/2} \leq h^{1/2}$, and (12.47), we obtain

$$(12.52) \quad \int_{|t| \leq T} \sum_{\mathfrak{c}} |Z_1^* Z_2^*| dt \ll N^\varepsilon T^{2+\varepsilon} \left(\frac{(\delta_4, q)}{q k_1 k_1^*} \right)^{1/2} \frac{\text{flrt}(\delta_2) \text{flrt}(\delta_5)^{3/2}}{\sqrt{\delta_2 \delta_5}} \sqrt{k_1 k_1^*}.$$

Recall that $\delta_5 = [\delta_3, \delta_4]$, and that δ_4 is square-free. Therefore, $[\delta_3, \delta_4] = \delta_3 \frac{\delta_4}{(\delta_3, \delta_4)}$ where $(\delta_3, \frac{\delta_4}{(\delta_3, \delta_4)}) = 1$. Since flrt is multiplicative, and trivial on square-free numbers, this implies

$$(12.53) \quad \text{flrt}(\delta_5) = \text{flrt}(\delta_3) \text{flrt} \left(\frac{\delta_4}{(\delta_3, \delta_4)} \right) = \text{flrt}(\delta_3),$$

a simplification we will make repeatedly below. Applying (12.53) to (12.52) gives (11.18).

Combining (12.9) and (12.41) and specializing the variables, we obtain

$$(12.54) \quad \sum_{\mathfrak{c}} |Z_1^0 Z_2^0| \ll (q(1 + |t|))^\varepsilon \left(\frac{(\delta_4, q)}{q k_1 k_1^*} \right) \sum_{\mathfrak{c}} \frac{(f, [\delta_4, q k_1 k_1^*])}{(f, \frac{N}{f})^2 N'' s^{1/2} f_r f}.$$

Using (12.23) and summing over $u \pmod{(f, \frac{N}{f})}$, we obtain

$$(12.55) \quad \sum_{\mathfrak{c}} \frac{(f, [\delta_4, q k_1 k_1^*])}{(f, \frac{N}{f})^2 N'' s^{1/2} f_r f} \leq \frac{1}{N} \sum_{f|N} \frac{(f, [\delta_4, q k_1 k_1^*])}{s^{1/2} f_r}.$$

Using the coprimality conditions, we have $[\delta_4, q k_1 k_1^*] = k_1 k_1^* [\delta_4, q]$, which in turn divides $k_1 k_1^* [\delta_5, q]$. Now k_1 is in the s -part of the level (since $s = \delta_2$ and $k_1 | \delta_2$), while we also have $(q, \delta_2) = 1$ by (7.8) so that $[\delta_4, q]$ is coprime to s , and hence f_s . Now we may see that the sum above factors as

$$\frac{1}{N} \sum_{f|N} \frac{(f, [\delta_4, q k_1 k_1^*])}{s^{1/2} f_r} \leq \left(\sum_{f_r | r} \frac{(f_r, [\delta_4, q])}{r f_r} \right) \left(\sum_{f_s | s} \frac{(f_s, k_1 k_1^*)}{s^{3/2}} \right).$$

Using $(f_r, [\delta_4, q]) \leq f_r$, $(f_s, k_1 k_1^*) \leq f_s^{1/2} \sqrt{k_1 k_1^*} \leq \sqrt{s k_1 k_1^*}$ leads immediately to (11.19).

Finally, we examine the two cross terms. From (12.22) and (12.41), and using simplifications as in the above cases, we have

$$(12.56) \quad \int_{|t| \leq T} \sum_{\mathfrak{c}} |Z_1^* Z_2^0| dt \ll \left(\frac{(\delta_4, q)}{q k_1 k_1^*} \right) q^\varepsilon T^{1+\varepsilon} \sum_{f | \delta_2 \delta_5} \frac{(f, \frac{\delta_2 \delta_5}{f})}{\delta_2 \delta_5 \sqrt{f}} (f, k_1 k_1^* [\delta_4, q]).$$

The inner sum factors as

$$(12.57) \quad \left(\sum_{g | \delta_2} \frac{(g, \frac{\delta_2}{g})}{\delta_2 \sqrt{g}} (g, k_1 k_1^*) \right) \left(\sum_{h | \delta_5} \frac{(h, \frac{\delta_5}{h})}{\delta_5 \sqrt{h}} (h, [\delta_4, q]) \right).$$

Using $(g, k_1 k_1^*) \leq \sqrt{g k_1 k_1^*}$, $(h, [\delta_4, q]) \leq h$, and (12.49), then

$$(12.58) \quad \sum_{f | \delta_2 \delta_5} \frac{(f, \frac{\delta_2 \delta_5}{f})}{\delta_2 \delta_5 \sqrt{f}} (f, k_1 k_1^* [\delta_4, q]) \ll q^\varepsilon \frac{\sqrt{\text{flrt}(\delta_5)} \text{flrt}(\delta_2)}{\delta_2 \sqrt{\delta_5}} \sqrt{k_1 k_1^*}.$$

Using (12.53), (11.20) follows.

Similarly, combining (12.9) and (12.45), we have

$$(12.59) \quad \int_{|t| \leq T} \sum_{\mathfrak{c}} |Z_1^0 Z_2^*| dt \ll \left(\frac{(\delta_4, q)}{q k_1 k_1^*} \right)^{1/2} q^\varepsilon T^{1+\varepsilon} \sum_{f|N} \frac{f(f, \frac{N}{f}) (k_1 k_1^* \frac{\delta_4 q}{(\delta_4, q)}, f)^{1/2}}{f_r N \sqrt{s f}}.$$

Following the discussion of the $Z_1^0 Z_2^0$ case (recall $r = \delta_5$ and $s = \delta_2$), the inner sum over f factors as

$$(12.60) \quad \left(\sum_{g|\delta_2} \frac{(g, \frac{\delta_2}{g})(k_1 k_1^*, g)^{1/2} g^{1/2}}{\delta_2^{3/2}} \right) \left(\sum_{h|\delta_5} \frac{(h, \frac{\delta_5}{h})(\frac{\delta_4 q}{(\delta_4, q)}, h)^{1/2}}{\delta_5 \sqrt{h}} \right)$$

Using (12.49) and (12.53), we have that this is

$$(12.61) \quad \ll q^\varepsilon \sqrt{k_1 k_1^*} \frac{\sqrt{\text{flrt}(\delta_2)} \text{flrt}(\delta_3)}{\delta_2 \delta_5}.$$

Hence we obtain (11.21), (in fact, with a slightly better power of $\text{flrt}(\delta_2)$).

13. ZERO TERMS

13.1. Overview. In this section, we analyze the contribution to \mathcal{S} from the terms with some $p_i = 0$. Recall the original expression for \mathcal{S}''' from (7.11), and Theorem 8.1.

Let us write

$$\mathcal{S}''' = \left(\sum_P \mathcal{T}_P \right) + \mathcal{S}_{0,0,0}''' + \mathcal{S}_{0,0}''' + \mathcal{S}_0''',$$

where \mathcal{T}_P corresponds to the terms with all $p_i \neq 0$, $\mathcal{S}_{0,0,0}'''$ corresponds to the terms with all three $p_i = 0$, $\mathcal{S}_{0,0}'''$ corresponds to the terms with exactly two $p_i = 0$, and finally \mathcal{S}_0''' has the terms with exactly one $p_i = 0$. Recall that the sum over P is the sum over the dyadic partition of unity. The partition is mainly beneficial for estimating \mathcal{T}_P , and we usually wish to remove the partition as much as possible when estimating the zero terms.

Applying the additional summations that led from \mathcal{S} to \mathcal{S}''' (see (7.10), (7.6), (7.1) or alternatively (13.8) and (13.9) below), we likewise define $\mathcal{S}_{0,0,0}$, $\mathcal{S}_{0,0}$, and \mathcal{S}_0 . Implicit in the definition of these quantities is that prior to the definition of \mathcal{S}''' , we applied a partition of unity. When it is necessary to emphasize this, we may write $\mathcal{S}_{0,0,0}^{(T)}$ where T stands for the tuple $(M_1, M_2, C, N_1, N_2, N_3, K)$, and likewise for $\mathcal{S}_{0,0}$ and \mathcal{S}_0 . Then $\sum_T \mathcal{S}_{0,0,0}^{(T)}$ represents the quantity after re-assembling the partition. For ease of notation we may on occasion drop the superscript T .

Our primary goal is to show

Theorem 13.1. *With an appropriate choice of $G_i(s)$ in the approximate functional equations, we have*

$$\sum_T \mathcal{S}_{0,0,0}^{(T)} \ll q^\varepsilon.$$

We will show the same bounds for $\mathcal{S}_{0,0}$ and \mathcal{S}_0 . We shall make extensive use of our assumption that

$$(13.1) \quad G_i(1/2) = 0.$$

Next we specialize Lemma 8.2 to degenerate p_i .

Lemma 13.2. *Let $(\alpha, k) = 1$. If some p_i is zero, then $A(p_1, p_2, p_3; \alpha; k)$ does not depend on α . Furthermore, we have*

$$(13.2) \quad \frac{1}{k} A(0, 0, 0; \alpha; k) = (\text{Id} * \phi)(k),$$

where $*$ indicates Dirichlet convolution, $\text{Id}(n) = n$, and ϕ is Euler's ϕ -function.

This is a short calculation, so we omit the proof.

13.2. The case with all $p_i = 0$. The case with $p_1 = p_2 = p_3 = 0$ is surprisingly delicate. It turns out that trivially bounding these terms leads only to $\mathcal{S}_{0,0,0} \ll q^{\frac{1}{4}+\varepsilon}$. Therefore, we have to make use of some further cancellation.

For notational simplicity, let us write $A(0, 0, 0; *, k'_0) = A(k'_0)$ (it is independent of $*$) and $B(0, 0, 0; k'_0) = B(k'_0)$. We then have

$$(13.3) \quad \mathcal{S}_{0,0,0}''' = \sum_{\substack{(c_0, g_0 m'_1)=1 \\ c_0 \equiv 0 \pmod{q k_1 k_1^*}}} \frac{1}{c_0} \sum_{\substack{(k'_0, \delta_2 c_0)=1 \\ k'_0 \equiv 0 \pmod{\frac{c_2}{(g_0, c_2)}}}} \frac{1}{k'_0{}^3} A(k'_0) B(k'_0).$$

The function B depends on a choice of a partition of unity in the n_1, n_2, n_3 variables (as well as c, k, m_1, m_2 , but here the focus is on the n_i). Our next goal is to recombine the partitions of unity in the dyadic numbers N_1, N_2, N_3 .

13.3. Recombining partitions of unity. We write the weight function more explicitly. Say

$$J(n_1, n_2, n_3) = J_*(n_1 n_2 n_3, a, m'_1, c_0, g_0 k'_0, c_2, k_1) F_a \left(\frac{n_1}{\sqrt{q}}, \frac{n_2}{\sqrt{q}}, \frac{n_3}{\sqrt{q}} \right) \frac{\omega \left(\frac{n_1}{N_1}, \frac{n_2}{N_2}, \frac{n_3}{N_3} \right)}{\sqrt{n_1 n_2 n_3}},$$

where

$$J_*(n, \cdot) = e \left(-\frac{nam_1}{ck} \right) \int_0^\infty e \left(\frac{-kt}{c} \right) J_{\kappa-1} \left(\frac{4\pi\sqrt{m_1 nat}}{c} \right) w_{M_2}(t, \cdot) \frac{dt}{\sqrt{t}}.$$

Here the weight function w_{M_2} is a dyadic partition of unity in the m_1, m_2, c , and k variables times $V(m_1 m_2 / q)$. The function J_* has $n = n_1 n_2 n_3$ appear as a block.

By (8.2), we have (introducing subscripts on B now to re-emphasize the choice of the partition of unity)

$$(13.4) \quad B_{N_1, N_2, N_3}(k'_0) = \iiint_{(\mathbb{R}^+)^3} \frac{J_*(e_1 e_2 \delta_1 t_1 t_2 t_3, a, m'_1, c_0, g_0 k'_0, c_2, k_1)}{\sqrt{\delta_1 e_1 e_2}} \\ \times F_a \left(\frac{t_1 r_1 e_1}{\sqrt{q}}, \frac{t_2 r_2 e_2}{\sqrt{q}}, \frac{t_3 r_3}{\sqrt{q}} \right) \omega \left(\frac{t_1 e_1 r_1}{N_1}, \frac{t_2 e_2 r_2}{N_2}, \frac{t_3 r_3}{N_3} \right) \frac{dt_1 dt_2 dt_3}{\sqrt{t_1 t_2 t_3}}.$$

The c, k, m_1, m_2 partitions are implicit in the definition of J_* .

Summing over all dyadic numbers $N_1, N_2, N_3 \geq 2^{-1/2}$, we obtain that

$$(13.5) \quad \sum_{2^{-1/2} \leq N_1, N_2, N_3 \text{ dyadic}} B_{N_1, N_2, N_3}(k'_0) \\ = \iiint_{(\mathbb{R}^+)^3} \frac{J_*(t_1 t_2 t_3, a, m'_1, c_0, g_0 k'_0, c_2, k_1)}{\delta_1 e_1 e_2} F_a \left(\frac{t_1}{\sqrt{q}}, \frac{t_2}{\sqrt{q}}, \frac{t_3}{\sqrt{q}} \right) W(t_1, t_2, t_3) \frac{dt_1 dt_2 dt_3}{\sqrt{t_1 t_2 t_3}},$$

where $W(t_1, t_2, t_3) = \sum_{2^{-1/2} \leq N_1, N_2, N_3 \text{ dyadic}} \omega(t_1/N_1, t_2/N_2, t_3/N_3)$. Note that the function $1 - W(t_1, t_2, t_3)$ is 0 if $t_i \geq 1$ for all i . It is a slightly subtle point that summing over the dyadic partition does not give $W(t_1, t_2, t_3) = 1$ for all $t_i > 0$.

Our immediate goal is to replace the W function by 1, and estimate the error. The basic idea is that $1 - W(t_1, t_2, t_3)$ should save a factor $q^{1/4}$ from the fact that at least one of the t_i is ≤ 1 , in place of $q^{1/2+\varepsilon}$. Here this numerology comes from that $\int_1^{q^{1/2}} t^{-1/2} dt \asymp q^{1/4}$, but

$\int_0^1 t^{-1/2} dt \ll 1$. In light of the claim that the trivial bound on $\mathcal{S}_{0,0,0}$ leads to $O(q^{1/4+\varepsilon})$, one naturally expects that this reasoning should lead to an acceptable final bound. Our next order of business is to confirm this expectation.

Lemma 13.3. *Let*

$$B_\Delta(k'_0) = \iiint_{(\mathbb{R}^+)^3} \frac{J_*(t_1 t_2 t_3, a, m'_1, c_0, g_0 k'_0, c_2, k_1)}{\delta_1 e_1 e_2} \times F_a\left(\frac{t_1}{\sqrt{q}}, \frac{t_2}{\sqrt{q}}, \frac{t_3}{\sqrt{q}}\right) (1 - W(t_1, t_2, t_3)) \frac{dt_1 dt_2 dt_3}{\sqrt{t_1 t_2 t_3}}.$$

Let \mathcal{S}_Δ''' be as in (13.3) but with B replaced with B_Δ . Then

$$(13.6) \quad \mathcal{S}_\Delta''' \ll q^\varepsilon \frac{(g_0, c_2)}{C c_2 k_1 k_1^*} \frac{m_1^{1/2} M_2}{a^{3/2} \delta_1 e_1 e_2}.$$

Proof. Notice that the support of $1 - W(t_1, t_2, t_3)$ is essentially included in the union of domains where one of the variables is in $[0, 1]$ and the other two are restricted to $[0, q^{\frac{1}{2}+\varepsilon}/a]$. The bound on the other two variables comes from the dropoff due to the function $F_{a, \sqrt{q}}(t_1, t_2, t_3)$.

Using only the trivial bound $J_{\kappa-1}(x) \ll x$ and $|I| = |J_*|$, we derive from (6.8) (which we bound trivially) that

$$(13.7) \quad |J_*(t_1 t_2 t_3, a m'_1, c_0, g_0 k'_0, c_2, k_1)| \ll \frac{M_2 \sqrt{m_1 a t_1 t_2 t_3}}{C}.$$

Therefore, using the above restrictions on the size of the t_i , and (3.6) we derive

$$B_\Delta(k'_0) \ll q^\varepsilon \frac{q m_1^{1/2} M_2}{a^{3/2} \delta_1 e_1 e_2 C}.$$

For the arithmetical part, we have

$$\frac{1}{k_0'^3} A(k'_0) \ll \frac{\tau(k'_0)}{k'_0}.$$

Hence

$$\mathcal{S}_\Delta''' \ll q^\varepsilon \sum_{\substack{(c_0, g_0 m'_1)=1 \\ c_0 \equiv 0 \pmod{q k_1 k_1^*}}} \frac{1}{c_0} \sum_{\substack{(k'_0, \delta_2 c_0)=1 \\ k'_0 \equiv 0 \pmod{\frac{c_2}{(g_0, c_2)}}}} \frac{q m_1^{1/2} M_2}{a^{3/2} \delta_1 e_1 e_2 C} \frac{\tau(k'_0)}{k'_0},$$

which quickly leads to (13.6). □

Recall that

$$(13.8) \quad \mathcal{S}_{0,0,0}'' = \sum_{\substack{g_0 | e_1 e_2 \delta_1 a m'_1 \\ g_0 \equiv 0 \pmod{d}}} \mathcal{S}_{0,0,0}''', \quad \mathcal{S}_{0,0,0}' = \sum_{r_1 r_2 r_3 = \delta_1} \sum_{e_1 | r_2 r_3} \sum_{e_2 | r_3} \mu(e_1) \mu(e_2) \mathcal{S}_{0,0,0}'' ,$$

and

$$(13.9) \quad \mathcal{S}_{0,0,0} = \sum_{(a, q)=1} \frac{\mu(a)}{a^{3/2}} \sum_{c_2} \frac{1}{c_2^{3/2}} \sum_{d | c_2} d \mu(c_2/d) \sum_{k_1} k_1^{1/2} \sum_{m'_1} \frac{1}{\sqrt{m'_1}} \mathcal{S}_0'.$$

Let $\mathcal{S}''_{\Delta}, \mathcal{S}'_{\Delta}$ and \mathcal{S}_{Δ} be defined similarly. Using Lemma 13.3 and $(g_0, c_2) \leq c_2$ then implies that

$$\mathcal{S}_{\Delta} \ll q^{\varepsilon} \sum_a \frac{1}{a^{3/2}} \sum_{c_2} \frac{1}{c_2^{3/2}} \sum_{d|c_2} d \sum_{k_1} k_1^{1/2} \sum_{m'_1} \frac{1}{\sqrt{m'_1}} \frac{(m'_1 k_1 c_2)^{1/2} M_2}{C k_1 k_1^* a^{3/2}}.$$

Using $m_1 = m'_1 k_1 c_2 \ll \frac{q^{1+\varepsilon}}{M_2}$, and $C \gg q$, we obtain

$$\mathcal{S}_{\Delta} \ll \frac{q^{1+\varepsilon}}{C} \ll q^{\varepsilon}.$$

Define $\overline{\mathcal{S}}''_{0,0,0}$ to be the same as $\mathcal{S}'''_{0,0,0}$ but with W replaced by 1, so that

$$\mathcal{S}'''_{0,0,0} = \overline{\mathcal{S}}'''_{0,0,0} + \mathcal{S}'''_{\Delta},$$

and similarly for $\overline{\mathcal{S}}''_{0,0,0}$, etc. To show Theorem 13.1, we therefore need to show $\sum_T \overline{\mathcal{S}}_{0,0,0} \ll q^{\varepsilon}$.

13.4. The function $B(k'_0)$. From now on, we let $\overline{B}(k'_0)$ be the function obtained from the right hand side of (13.5) after replacing W by 1, and summing over the dyadic variables C and K . This has the shape

$$(13.10) \quad \overline{B}(k'_0) = \iiint_{(\mathbb{R}^+)^3} \frac{J_*(t_1 t_2 t_3, a, m'_1, c_0, g_0 k'_0, c_2, k_1)}{\delta_1 e_1 e_2} F_a \left(\frac{t_1}{\sqrt{q}}, \frac{t_2}{\sqrt{q}}, \frac{t_3}{\sqrt{q}} \right) \frac{dt_1 dt_2 dt_3}{\sqrt{t_1 t_2 t_3}},$$

where we did not give a new name to J_* after summing over C and K . This is the relevant function for evaluating $\overline{\mathcal{S}}_{0,0,0}$.

Proposition 13.4. *Denote*

$$(13.11) \quad \mathcal{H}(s, w, u, \kappa) = (-1)^{\frac{\kappa}{2}} \frac{(2\pi)^{s+w+u-1} \Gamma(s+w+u) \Gamma(\frac{\kappa}{2} - w - u) \Gamma(\frac{\kappa}{2} - s)}{\Gamma(\frac{\kappa}{2} + s) \Gamma(\frac{\kappa}{2} + w + u)}.$$

Here \mathcal{H} is holomorphic in the region

$$\operatorname{Re}(s), \operatorname{Re}(w+u) < \frac{\kappa}{2}, \quad \text{and} \quad \operatorname{Re}(s+w+u) > 0,$$

with polynomial growth in $\operatorname{Im}(s)$, $\operatorname{Im}(w)$, and $\operatorname{Im}(u)$ in vertical strips. With this notation,

$$\begin{aligned} \overline{B}(k'_0) &= \frac{1}{\delta_1 e_1 e_2} \int_{(1-\varepsilon)} \frac{\gamma(1/2 + s, \kappa)^3 G(s)^3}{\gamma(1/2, \kappa)^3 s^3} \zeta_q(1+2s)^3 q^{3s/2} a^{-3s} \\ &\quad \int_{(1-2\varepsilon)} \tilde{V}(w) \left(\frac{m_1}{q} \right)^{-w} \int_{(-2\varepsilon)} M_2^u \tilde{\omega}(u, \cdot) \frac{1}{k^{s-w-u}} \frac{(am_1)^{s-1/2}}{c^{s+w+u-1}} \mathcal{H}(s, w, u, \kappa) \frac{dudwds}{(2\pi i)^3}, \end{aligned}$$

where $k = g_0 k'_0 k_1$, $m_1 = m'_1 k_1 c_2$, $c = c_0 c_2$, and $\tilde{\omega}(u, \cdot) = \tilde{\omega}(u) \omega(m_1/M_1)$.

Proof. Note that in (13.10), the factor $t_1 t_2 t_3$ shows up as a block in both J and the denominator. Letting $y = t_1 t_2 t_3$ (viewing t_2 and t_3 as fixed), we have

$$\overline{B}(k'_0) = \int_0^\infty \frac{J_*(y, a, m'_1, c_0, g_0 k'_0, c_2, k_1)}{\delta_1 e_1 e_2} \int_0^\infty \int_0^\infty F_a \left(\frac{y/t_2 t_3}{\sqrt{q}}, \frac{t_2}{\sqrt{q}}, \frac{t_3}{\sqrt{q}} \right) \frac{dt_2}{t_2} \frac{dt_3}{t_3} \frac{dy}{\sqrt{y}}.$$

We first claim that

$$\int_0^\infty \int_0^\infty F_a \left(\frac{y/t_2 t_3}{\sqrt{q}}, \frac{t_2}{\sqrt{q}}, \frac{t_3}{\sqrt{q}} \right) \frac{dt_2}{t_2} \frac{dt_3}{t_3} = \int_{(2)} \frac{q^{\frac{3s}{2}}}{(a^3 y)^s} \frac{\gamma(\frac{1}{2} + s, \kappa)^3 G(s)^3}{\gamma(\frac{1}{2}, \kappa)^3 s^3} \zeta_q(1 + 2s)^3 \frac{ds}{2\pi i}.$$

This is an exercise with Mellin inversion, directly using the definition (3.5). Secondly, we claim

$$(13.12) \quad \int_0^\infty J_*(y, a, m'_1, c_0, g_0 k'_0, c_2, k_1) y^{-\frac{1}{2}-s} dy \\ = \int_{(1-2\varepsilon)} \tilde{V}(w) \left(\frac{m_1}{q} \right)^{-w} \int_{(-2\varepsilon)} M_2^u \tilde{\omega}(u, \cdot) \frac{1}{k^{s-w-u}} \frac{(am_1)^{s-1/2}}{c^{s+w+u-1}} \mathcal{H}(s, w, u, \kappa) \frac{dudw}{(2\pi i)^2}.$$

Putting these two claims together then completes the proof.

Now we show (13.12). From (7.3) and (6.8), and summing over the C and K partitions, we have

$$J_*(y, a, m'_1, c_0, g_0 k'_0, c_2, k_1) = e \left(-\frac{yam'_1}{c_0 g_0 k'_0} \right) I(m'_1 k_1 c_2, g_0 k'_0 k_1, ya, c_0 c_2) \\ = \int_0^\infty e \left(-\frac{yam'_1}{c_0 g_0 k'_0} \right) e \left(\frac{-g_0 k'_0 k_1 t}{c_0 c_2} \right) J_{\kappa-1} \left(\frac{4\pi \sqrt{m'_1 k_1 c_2 y a t}}{c_0 c_2} \right) V_1 \left(\frac{m'_1 k_1 c_2 t}{q} \right) \omega_{M_2}(t, \cdot) \frac{dt}{\sqrt{t}} \\ = \int_0^\infty e \left(-\frac{yam_1}{ck} \right) e \left(\frac{-kt}{c} \right) J_{\kappa-1} \left(\frac{4\pi \sqrt{m_1 y a t}}{c} \right) V_1 \left(\frac{m_1 t}{q} \right) \omega_{M_2}(t, \cdot) \frac{dt}{\sqrt{t}},$$

where for simplicity in the final line above we have written the expression in terms of the earlier variable names, and where $\omega_{M_2}(t, \cdot) = \omega(t/M_2)\omega(m_1/M_1)$ (since we have summed over C and K , as well as the N_i). Therefore, (13.12) equals

$$\int_0^\infty \int_0^\infty e \left(-\frac{yam_1}{ck} \right) e \left(\frac{-kt}{c} \right) J_{\kappa-1} \left(\frac{4\pi \sqrt{m_1 y a t}}{c} \right) V_1 \left(\frac{m_1 t}{q} \right) \omega_{M_2}(t, \cdot) \frac{dt}{\sqrt{t}} y^{-s} \frac{dy}{\sqrt{y}}.$$

Change variables by $y = z/t$ (after interchanging the order of integration), giving that (13.12) equals

$$\int_0^\infty \int_0^\infty e \left(-\frac{zam_1}{ckt} \right) e \left(\frac{-kt}{c} \right) J_{\kappa-1} \left(\frac{4\pi \sqrt{m_1 a z}}{c} \right) V_1 \left(\frac{m_1 t}{q} \right) \omega_{M_2}(t, \cdot) t^s \frac{dt}{t} z^{-s} \frac{dz}{\sqrt{z}}.$$

Rewriting V_1 and ω_{M_2} in terms of their Mellin transforms, we have that (13.12) is

$$(13.13) \quad \int_{(1-2\varepsilon)} \tilde{V}(w) \left(\frac{m_1}{q} \right)^{-w} \int_{(-\varepsilon)} M_2^u \tilde{\omega}(u, \cdot) \mathcal{I} \frac{dudw}{(2\pi i)^2},$$

where \mathcal{I} is shorthand for

$$(13.14) \quad \mathcal{I} = \int_0^\infty J_{\kappa-1} \left(\frac{4\pi \sqrt{m_1 a z}}{c} \right) z^{-s} \left(\int_0^\infty e \left(-\frac{m_1 a z}{ckt} \right) e \left(\frac{-kt}{c} \right) t^{s-w-u} \frac{dt}{t} \right) \frac{dz}{\sqrt{z}}.$$

We will derive an explicit formula for \mathcal{I} by consulting tables of integrals.

Lemma 13.5. *For $|\operatorname{Re}(s - w - u)| < 1$, we have*

$$\begin{aligned} \int_0^\infty e\left(-\frac{m_1 a z}{c k t}\right) e\left(\frac{-k t}{c}\right) t^{s-w-u} \frac{dt}{t} \\ = -i\pi \left(\frac{\sqrt{m_1 a z}}{k}\right)^{s-w-u} e^{-\pi i \frac{s-w-u}{2}} H_{s-w-u}^{(2)}\left(\frac{4\pi\sqrt{m_1 a z}}{c}\right). \end{aligned}$$

Here

$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z)$$

is the Hankel function of the second kind.

Proof. This follows from [14, (3.871.1), (3.871.2)], or formulas (17) and (36) in [11, Section 6.5]. \square

Even though the original calculation requires $|\operatorname{Re}(s - w - u)| < 1$ for convergence, note that the Hankel function $H_\nu^{(2)}$ is an analytic function of ν and hence we may move our lines of integration in s, w and u to any location without encountering any poles from the Hankel function.

Inserting this evaluation into (13.14), we have

$$\mathcal{I} = -i\pi e^{-\pi i \frac{s-w-u}{2}} \int_0^\infty \left(\frac{\sqrt{m_1 a z}}{k}\right)^{s-w-u} H_{s-w-u}^{(2)}\left(\frac{4\pi\sqrt{m_1 a z}}{c}\right) J_{\kappa-1}\left(\frac{4\pi\sqrt{m_1 a z}}{c}\right) z^{\frac{1}{2}-s} \frac{dz}{z}.$$

The Bessel and Hankel functions have the same argument, which is quite pleasant. By changing variables, we have

$$(13.15) \quad \mathcal{I} = \frac{-ie^{-\pi i \frac{s-w-u}{2}} (am_1)^{s-1/2} (4\pi)^{s+w+u}}{2k^{s-w-u} c^{s+w+u-1}} \int_0^\infty H_{s-w-u}^{(2)}(z) J_{\kappa-1}(z) z^{1-s-w-u} \frac{dz}{z}.$$

The z -integral may be evaluated in closed form.

Lemma 13.6. *For $\operatorname{Re}(\pm\nu - \mu) < \operatorname{Re}(\lambda) < 1$, we have*

$$(13.16) \quad \int_0^\infty H_\nu^{(2)}(x) J_\mu(x) x^\lambda \frac{dx}{x} = \frac{i2^{\lambda-1} \Gamma(1-\lambda) \Gamma(\frac{\nu+\mu+\lambda}{2}) \Gamma(\frac{\mu-\nu+\lambda}{2})}{\pi \Gamma(\frac{\nu+\mu-\lambda}{2} + 1) \Gamma(\frac{\mu-\nu-\lambda}{2} + 1)} e^{-\frac{\pi}{2}i(\mu-\nu+\lambda)}.$$

Proof. We may calculate this using formulas (33) and (36) in [11, Section 6.8] (but note (36) is missing a $\Gamma(1-\lambda)$ term), and simplifying using gamma function identities. \square

Substituting

$$\lambda = 1 - s - w - u, \quad \nu = s - w - u, \quad \text{and} \quad \mu = \kappa - 1,$$

the region of convergence corresponds to

$$(13.17) \quad \operatorname{Re}(s), \operatorname{Re}(w + u) < \frac{\kappa}{2} \quad \text{and} \quad \operatorname{Re}(s + w + u) > 0,$$

which are satisfied by the lines of integration given in (13.13). Furthermore,

$$\mathcal{I} = \frac{(-i)^\kappa e^{\pi i \frac{s+w+u}{2}} (am_1)^{s-1/2} (2\pi)^{s+w+u-1} \Gamma(s+w+u) \Gamma(\frac{\kappa}{2} - w - u) \Gamma(\frac{\kappa}{2} - s)}{k^{s-w-u} c^{s+w+u-1} \Gamma(\frac{\kappa}{2} + s) \Gamma(\frac{\kappa}{2} + w + u)},$$

giving

$$(13.18) \quad \mathcal{I} = \frac{\mathcal{H}(s, w, u, \kappa) (am_1)^{s-1/2}}{k^{s-w-u} c^{s+w+u-1}}.$$

An application of Stirling's approximation shows the growth in $\text{Im}(s)$, $\text{Im}(w)$ and $\text{Im}(u)$ is bounded by a polynomial.

Inserting this formula for \mathcal{I} into (13.13), we complete the proof of Proposition 13.4. \square

13.5. Bounding the zero term. Now let us recall that $k = g_0 k'_0 k_1$, $m_1 = m'_1 k_1 c_2$ and $c = c_0 c_2$. We will substitute the evaluation of \overline{B} into $\overline{\mathcal{S}}'''_{0,0,0}$ which was defined as (13.3) (with the partition of unity removed). This gives

$$(13.19) \quad \overline{\mathcal{S}}'''_{0,0,0} = \frac{1}{\delta_1 e_1 e_2} \sum_{\substack{(c_0, g_0 m'_1)=1 \\ c_0 \equiv 0 \pmod{q k_1 k_1^*}}} \frac{1}{c_0} \sum_{\substack{(k'_0, \delta_2 c_0)=1 \\ k'_0 \equiv 0 \pmod{\frac{c_2}{(g_0, c_2)}}}} \frac{A(k'_0)}{k_0'^3} \int_{(1-\varepsilon)} \frac{\gamma(1/2 + s, \kappa)^3 G(s)^3}{\gamma(1/2, \kappa)^3 s^3} \\ \zeta_q(1 + 2s)^3 q^{3s/2} a^{-3s} \int_{(1-2\varepsilon)} \tilde{V}(w) \left(\frac{q}{m'_1 k_1 c_2} \right)^w \int_{(-2\varepsilon)} M_2^u \tilde{\omega}(u, \cdot) \\ \frac{\mathcal{H}(s, w, u, \kappa)}{(g_0 k'_0 k_1)^{s-w-u}} \frac{(am'_1 k_1 c_2)^{s-1/2}}{(c_0 c_2)^{s+w+u-1}} \frac{dudwds}{(2\pi i)^3}.$$

Next examine the Dirichlet series

$$\mathcal{Z}(s, w, u) = \zeta_q(1 + 2s)^3 \sum_{\substack{(c_0, g_0 m'_1)=1 \\ c_0 \equiv 0 \pmod{q k_1 k_1^*}}} \frac{1}{c_0^{s+w+u}} \sum_{\substack{(k'_0, \delta_2 c_0)=1 \\ k'_0 \equiv 0 \pmod{\frac{c_2}{(g_0, c_2)}}}} \frac{\frac{1}{k_0'^3} A(0, 0, 0; k'_0)}{k_0'^{s-w-u}}.$$

Using the evaluation $A(0, 0, 0; k'_0) = k'_0 \sum_{d|k'_0} d\phi(k'_0/d)$ and Möbius inversion to remove the condition $(k'_0, c_0) = 1$, one may derive

$$(13.20) \quad \mathcal{Z}(s, w, u) = \frac{\zeta(1 + s - w - u)^2 \zeta(s + w + u)}{c_2^{s-w-u+1} (q k_1 k_1^*)^{s+w+u}} (g_0, c_2)^{s-w-u+1} \Delta(s, w, u),$$

where $\Delta(s, w, u)$ is analytic for

$$(13.21) \quad \text{Re}(s) > 0, \quad \text{Re}(u + w) < 1 + \text{Re}(s), \quad \text{Re}(s + w + u) > 0,$$

and bounded by $O(q^\varepsilon)$ in that region. The sum defining $\mathcal{Z}(s, w, u)$ converges absolutely for $\text{Re}(s + w + u) > 1$ and $\text{Re}(w + u) < \text{Re}(s)$.

Inserting (13.20) into (13.19), we obtain

$$(13.22) \quad \overline{\mathcal{S}}'''_{0,0,0} = \int_{(1-\varepsilon)} \frac{\gamma(1/2 + s, \kappa)^3 G(s)^3}{\delta_1 e_1 e_2 \gamma(1/2, \kappa)^3 s^3} \frac{q^{3s/2}}{a^{3s}} \int_{(1-2\varepsilon)} \tilde{V}(w) \left(\frac{q}{m'_1 k_1 c_2} \right)^w \int_{(-2\varepsilon)} M_2^u \tilde{\omega}(u, \cdot) \\ \frac{\zeta(1 + s - w - u)^2 \zeta(s + w + u)}{c_2^{s-w-u+1} (q k_1 k_1^*)^{s+w+u}} (g_0, c_2)^{s-w-u+1} \frac{\Delta \mathcal{H}(s, w, u, \kappa)}{(g_0 k_1)^{s-w-u}} \frac{(am'_1 k_1 c_2)^{s-1/2}}{c_2^{s+w+u-1}} \frac{dudwds}{(2\pi i)^3}.$$

Now move the contour of integration in w to the line $\text{Re}(w) = 4\varepsilon$. In doing that note that we still have $\text{Re}(s + w + u) = 1 - \varepsilon + 4\varepsilon - \varepsilon = 1 + 2\varepsilon > 1$ and $\text{Re}(s - w - u + 1) = 1 - \varepsilon - 4\varepsilon + 2\varepsilon + 1 = 2 - 3\varepsilon > 1$, so we do not pass over any poles. Now move the line of integration in s to $\text{Re}(s) = 3\varepsilon$. By doing so, we pick up the residue from the simple pole at $s = 1 - w - u$.

The remaining integral. The contribution from the final integral to $\overline{\mathcal{S}}'''_{0,0,0}$ is at most

$$(13.23) \quad \ll \frac{(g_0, c_2) q^\varepsilon}{\delta_1 e_1 e_2 \sqrt{am'_1 k_1 c_2}}.$$

The contribution to $\overline{\mathcal{S}}_{0,0,0}$ from this part is then calculated (recall that $\delta_1 = k_1 d / (a, k_1 d)$ and $\delta_2 = e_1 e_2 \delta_1 a m'_1 / g_0$) to be at most

$$\begin{aligned} & \sum_a \frac{1}{a^{3/2}} \sum_{c_2} \frac{1}{c_2^{3/2}} \sum_{d|c_2} d \sum_{k_1} k_1^{1/2} \sum_{m'_1} \frac{1}{\sqrt{m'_1}} \sum_{r_1 r_2 r_3 = \delta_1} \sum_{e_1 | r_2 r_3} \sum_{e_2 | r_3} \sum_{g_0 | e_1 e_2 \delta_1 a m'_1} \frac{(g_0, c_2) q^\varepsilon}{\delta_1 e_1 e_2 \sqrt{a m'_1 k_1 c_2}} \\ & \ll q^\varepsilon \sum_a \frac{1}{a^2} \sum_{c_2} \frac{1}{c_2^2} \sum_{d|c_2} \sum_{k_1} \frac{(k_1 d, a)}{k_1} \sum_{m'_1} \frac{1}{m'_1} \sum_{e_1 | r_2 r_3} \sum_{e_2 | r_3} \frac{1}{e_1 e_2} \sum_{g_0 | e_1 e_2 \delta_1 a m'_1} (g_0, c_2), \end{aligned}$$

where all the summations may be truncated at some fixed power of q (cf. the convention in Section 11.6). Summing over everything trivially using $(g_0, c_2) \leq c_2$ shows that the integral contribution to $\overline{\mathcal{S}}_{0,0,0}$ is $O(q^\varepsilon)$.

The $s = 1 - w - u$ residue. This residue contributes to $\overline{\mathcal{S}}_{0,0,0}'''$ the following:

$$(13.24) \quad \int_{(4\varepsilon)} \frac{\gamma(3/2 - w - u, \kappa)^3 G(1 - w - u)^3}{\delta_1 e_1 e_2 \gamma(1/2, \kappa)^3 (1 - w - u)^3} \tilde{V}(w) \left(\frac{1}{m'_1 k_1 c_2} \right)^w \int_{(-2\varepsilon)} M_2^u \tilde{\omega}(u, \cdot) \\ \frac{q^{\frac{1-w-u}{2}}}{a^{3(1-w-u)}} \frac{\zeta(2 - 2w - 2u)^2}{c_2^{2-2w-2u} (k_1 k_1^*)} (g_0, c_2)^{2-2w-2u} \frac{\Delta \mathcal{H}(s, w, u, \kappa)}{(g_0 k_1)^{1-2w-2u}} (a m'_1 k_1 c_2)^{1/2-w-u} \frac{dudw}{(2\pi i)^2}.$$

Now move the line of integration in w to $\text{Re}(w) = 1 - \varepsilon$. This will pass over an apparent double pole of $\zeta(2 - 2w - 2u)$ but the triple zero of $G(1 - w - u)^3$ cancels it. Then by a trivial bound, we have that the residue is

$$(13.25) \quad \ll \frac{g_0 q^\varepsilon}{\delta_1 e_1 e_2 a^{1/2} c_2^{3/2} k_1^{3/2} k_1^* m_1^{3/2}}.$$

The contribution coming from the residue can be bounded as follows:

$$\sum_a \frac{1}{a^2} \sum_{c_2} \frac{1}{c_2^3} \sum_{d|c_2} d \sum_{k_1} \frac{1}{k_1 k_1^*} \sum_{m'_1} \frac{1}{m_1'^2} \sum_{r_1 r_2 r_3 = \delta_1} \frac{1}{\delta_1} \sum_{e_1 | r_2 r_3} \sum_{e_2 | r_3} \frac{1}{e_1 e_2} \sum_{g_0 | e_1 e_2 \delta_1 a m'_1} g_0.$$

Let us trivially bound $g_0 \leq e_1 e_2 \delta_1 a m'_1$. All the remaining sums are easily bounded, so this part is also $O(q^\varepsilon)$.

This completes the proof of Theorem 13.1.

13.6. One of the p_i is zero. This case is the easiest, since (as it turns out) we may bound everything trivially and obtain the desired bound $\mathcal{S}_0 \ll q^\varepsilon$.

The original sum is symmetric in p_1, p_2 and p_3 , so it suffices to estimate the terms with $p_3 = 0$, and $p_1, p_2 \neq 0$ (the expression for A from Lemma 8.2 may not appear symmetric in the p_i , but of course it must be due to the original definition (8.1)). Let $P_1, P_2 \neq 0$, put $P = (P_1, P_2, 0)$, and consider

$$\mathcal{S}_P''' = \sum_{\substack{(c_0, g_0 m'_1)=1 \\ c_0 \equiv 0 \pmod{q k_1 k_1^*}}} \frac{1}{c_0} \sum_{\substack{(k'_0, \delta_2 c_0)=1 \\ k'_0 \equiv 0 \pmod{\frac{c_2}{(g_0, c_2)}}}} \sum_{\substack{p_1 \lesssim P_1 \\ p_2 \lesssim P_2}} \frac{1}{k_0'^3} A(p_1, p_2, 0; k'_0) B(p_1, p_2, 0).$$

Here

$$A(p_1, p_2, 0; k'_0) = k'_0 \sum_{f | (p_2, k'_0)} f S(p_1, 0; k'_0 / f) \ll k_0'^{1+\varepsilon} (p_1 p_2, k'_0).$$

Note that we only need to consider the non-oscillatory cases for B , where B is given by (8.12), since in the oscillatory case all the p_i must be nonzero or else B is very small. Then

$$\mathcal{S}_P''' \ll \sum_{\substack{(c_0, g_0 m'_1)=1 \\ c_0 \equiv 0 \pmod{q k_1 k_1^*}}} \frac{1}{c_0} \sum_{\substack{(k'_0, \delta_2 c_0)=1 \\ k'_0 \equiv 0 \pmod{\frac{c_2}{(g_0, c_2)}}}} \frac{1}{k_0'^2} \sum_{p_1, p_2 \neq 0} \left(\frac{\sqrt{aMN}}{C} \right)^\delta \frac{\sqrt{M_2 N}}{h} (p_1 p_2, k'_0),$$

where recall $\delta = \kappa - 1 \geq 1$ in the pre-transition non-oscillatory range, and $\delta = -1$ in the post-transition range. Recall $P_1 P_2 \ll q^\varepsilon \frac{k_0'^2}{N_2' N_3'} \ll q^\varepsilon \frac{k_0'^2 h}{N_2 N_3}$. Therefore,

$$\mathcal{S}_P''' \ll q^\varepsilon \left(\frac{\sqrt{aMN}}{C} \right)^\delta \frac{\sqrt{M_2 N}}{N_2 N_3} \frac{K(g_0, c_2)}{g_0 k_1 c_2} \frac{1}{q k_1 k_1^*}.$$

It is then not difficult to see that

$$\mathcal{S}_P \ll q^\varepsilon \max_a \left(\frac{\sqrt{aMN}}{C} \right)^\delta \frac{\sqrt{M_2 N} K}{q N_2 N_3 \sqrt{a}} \sqrt{M_1}.$$

In the **Post-transition case**, this bound becomes

$$\mathcal{S}_P \ll q^\varepsilon \max_a \frac{M_1 N_1}{q} \ll q^\varepsilon.$$

The **Pre-transition, non-oscillatory case** leads to the same bound. Summing over the dyadic values of P gives $\mathcal{S}_0 \ll q^\varepsilon$, as desired.

13.7. Two of the p_i are zero. We finally consider the case where say $p_1 \neq 0$, and $p_2 = p_3 = 0$. This case leads to some new subtleties not present in the case with all $p_i = 0$. The first step is to extend the sum to all $p_1 \in \mathbb{Z}$, and then subtract back the term with $p_1 = 0$. We already showed with Theorem 13.1 that the term with all $p_i = 0$ is bounded in an acceptable way. After this, we apply Poisson summation backwards. The net effect is precisely the same as only applying Poisson summation in the n_2 - and n_3 -variables, and setting $p_2 = p_3 = 0$ (up to the term with all $p_i = 0$).

It is perhaps easiest to return to (7.11). Define \mathcal{Q} to be the term we get from this, after Poisson in n_2 and n_3 , and substitution of $p_2 = p_3 = 0$, so that

$$\mathcal{Q} = \sum_{\substack{(c_0, g_0 m'_1)=1 \\ c_0 \equiv 0 \pmod{q k_1 k_1^*}}} \frac{1}{c_0} \sum_{\substack{(k'_0, \delta_2 c_0)=1 \\ k'_0 \equiv 0 \pmod{\frac{c_2}{(g_0, c_2)}}}} \sum_{n_1 \geq 1} \frac{A^*(n_1; k'_0) B^*(n_1)}{\sqrt{n_1}},$$

where

$$A^*(n_1; k'_0) = \frac{1}{k_0'^2} \sum_{x_2, x_3 \pmod{k'_0}} e\left(\frac{\delta_2 n_1 x_2 x_3 \overline{c_0}}{k'_0}\right),$$

and

$$\begin{aligned} B^*(n_1) &= \frac{1}{\sqrt{\delta_1 e_1 e_2}} \int_0^\infty \int_0^\infty F_a\left(\frac{r_1 e_1 n_1}{\sqrt{q}}, \frac{r_2 e_2 t_2}{\sqrt{q}}, \frac{r_3 t_3}{\sqrt{q}}\right) \omega\left(\frac{n_1 e_1 r_1}{N_1}, \frac{t_2 e_2 r_2}{N_2}, \frac{t_3 r_3}{N_3}\right) \\ &\quad \times J_*(e_1 e_2 \delta_1 n_1 t_2 t_3, a, m'_1, c_0, g_0 k'_0, c_2, k_1) \frac{dt_2 dt_3}{\sqrt{t_2 t_3}}. \end{aligned}$$

Since some of the details are similar (and easier) than the case where all $p_i = 0$, we will be more brief in such occasions. We may evaluate A^* directly from the definition, using a similar method to the proof of Lemma 8.2, which gives

$$A^*(n_1; k'_0) = \frac{1}{k'_0} \sum_{f|k'_0} \varphi\left(\frac{k'_0}{f}\right) \delta(n_1 \equiv 0 \pmod{\frac{k'_0}{f}}).$$

We have $J_*(\dots) \ll M_2^{1/2}$, which follows from bounding $J_{\kappa-1}(x) \ll 1$, and so

$$B^*(n_1) \ll \left(\frac{M_2 N_2 N_3}{r_1 r_2 r_3 e_1 e_2 e_2 r_2 r_3} \right)^{1/2}.$$

In turn, this leads to the estimate

$$\mathcal{Q} \ll q^\varepsilon \frac{\sqrt{M_2 N}(g_0, c_2)}{\delta_1 e_1 e_2 q k_1 k_1^* c_2}.$$

Then the contribution to \mathcal{S} from \mathcal{Q} is seen to be $O(q^{-1+\varepsilon}(MN)^{1/2}) = O(q^{1/4+\varepsilon})$.

This numerology shows that we may remove the partition of unity at cost $O(q^\varepsilon)$ (essentially, restricting a t_2 - or t_3 -integral to be in the range $O(1)$ instead of $O(q^{1/2})$ saves $q^{1/4}$ over the previous bound). Since $n_1 \geq 1$ automatically, we may easily sum over the N_1 -partition (avoiding the analytic problems near the origin). Let us define $\overline{\mathcal{Q}}$ to be the sum obtained from this replacement, and $\overline{B}^*(n_1)$ to be the new integral. For ease of notation we shall not change the notation of the function J_* . Then by the change of variables $y = e_1 e_2 \delta_1 n_1 t_2 t_3$ (viewing t_3 as fixed), we have

$$\overline{B}^*(n_1) = \int_0^\infty \int_0^\infty \frac{J_*(y, a, m'_1, c_0, g_0 k'_0, c_2, k_1)}{\delta_1 e_1 e_2 \sqrt{n_1}} F_a\left(\frac{r_1 e_1 n_1}{\sqrt{q}}, \frac{\frac{y}{e_1 r_1 r_3 n_1 t_3}}{\sqrt{q}}, \frac{r_3 t_3}{\sqrt{q}}\right) \frac{dt_3}{t_3} \frac{dy}{\sqrt{y}}.$$

By an exercise with Mellin inversion, one may show

$$\begin{aligned} \int_0^\infty F_a\left(\frac{r_1 e_1 n_1}{\sqrt{q}}, \frac{\frac{y}{e_1 r_1 r_3 n_1 t_3}}{\sqrt{q}}, \frac{r_3 t_3}{\sqrt{q}}\right) \frac{dt_3}{t_3} &= \int_{(1)} \int_{(1)} \frac{\gamma(\frac{1}{2} + s_1, \kappa) G(s_1)}{\gamma(\frac{1}{2}, \kappa) s_1} \frac{\gamma(\frac{1}{2} + s, \kappa)^2 G(s)^2}{\gamma(\frac{1}{2}, \kappa)^2 s^2} \\ &\quad \times a^{-2s-s_1} \zeta_q(1 + s_1 + s)^2 \zeta_q(1 + 2s) \left(\frac{\sqrt{q}}{e_1 r_1 n_1}\right)^{s_1} \left(\frac{q e_1 r_1 n_1}{y}\right)^s \frac{ds_1 ds}{(2\pi i)^2}. \end{aligned}$$

Then using (13.12), we derive

$$\begin{aligned} \overline{B}^*(n_1) &= \int_{(1-2\varepsilon)} \frac{\tilde{V}(w)}{\delta_1 e_1 e_2 \sqrt{n_1}} \int_{(1)} \frac{\gamma(\frac{1}{2} + s_1, \kappa) G(s_1)}{\gamma(\frac{1}{2}, \kappa) s_1} \int_{(1)} \frac{\gamma(\frac{1}{2} + s, \kappa)^2 G(s)^2}{\gamma(\frac{1}{2}, \kappa)^2 s^2} \\ &\quad \int_{(0)} M_2^u \tilde{\omega}(u, \cdot) a^{-2s-s_1} \zeta_q(1 + s_1 + s)^2 \zeta_q(1 + 2s) \left(\frac{\sqrt{q}}{e_1 r_1 n_1}\right)^{s_1} (q e_1 r_1 n_1)^s \\ &\quad \left(\frac{q}{m_1}\right)^w \frac{\mathcal{H}(s, w, u, \kappa)}{k^{s-w-u}} \frac{(a m_1)^{s-1/2}}{c^{s+w+u-1}} \frac{du ds ds_1 dw}{(2\pi i)^4}, \end{aligned}$$

where recall that $k = g_0 k'_0 k_1$, $m_1 = m'_1 k_1 c_2$, and $c = c_0 c_2$. Moreover, as in Section 13.5, we have $\tilde{\omega}(u, \cdot) = \tilde{\omega}(u) \omega(m_1/M_1)$, since we have summed over N_1, N_2, N_3, C , and K .

Applying these changes of variables, and inserting this into the definition of $\overline{\mathcal{Q}}$, we obtain

$$\begin{aligned} \overline{\mathcal{Q}} = & \frac{1}{\delta_1 e_1 e_2} \sum_{\substack{(c_0, g_0 m'_1)=1 \\ c_0 \equiv 0 \pmod{q k_1 k_1^*}}} \frac{1}{c_0} \sum_{\substack{(k'_0, \delta_2 c_0)=1 \\ k'_0 \equiv 0 \pmod{\frac{c_2}{(g_0, c_2)}}}} \frac{1}{k'_0} \sum_{f|k'_0} \varphi\left(\frac{k'_0}{f}\right) \sum_{n_1 \equiv 0 \pmod{\frac{k'_0}{f}}} \frac{1}{n_1} \\ & \int_{(1-2\varepsilon)} \tilde{V}(w) \int_{(1)} \frac{\gamma(\frac{1}{2} + s_1, \kappa) G(s_1)}{\gamma(\frac{1}{2}, \kappa) s_1} \int_{(1-\varepsilon)} \frac{\gamma(\frac{1}{2} + s, \kappa)^2 G(s)^2}{\gamma(\frac{1}{2}, \kappa)^2 s^2} \\ & \int_{(0)} M_2^u \tilde{\omega}(u) a^{-2s-s_1} \zeta_q(1+s_1+s)^2 \zeta_q(1+2s) \left(\frac{\sqrt{q}}{e_1 r_1 n_1}\right)^{s_1} (q e_1 r_1 n_1)^s \\ & \left(\frac{q}{m'_1 k_1 c_2}\right)^w \frac{\mathcal{H}(s, w, u, \kappa)}{(g_0 k'_0 k_1)^{s-w-u}} \frac{(a m'_1 k_1 c_2)^{s-1/2}}{(c_0 c_2)^{s+w+u-1}} \frac{dudsd s_1 dw}{(2\pi i)^4}. \end{aligned}$$

With the displayed lines of integration, all the outer sums converge absolutely. Indeed, we have

$$\begin{aligned} (13.26) \quad & \sum_{\substack{(c_0, g_0 m'_1)=1 \\ c_0 \equiv 0 \pmod{q k_1 k_1^*}}} \frac{1}{c_0^{s+w+u}} \sum_{\substack{(k'_0, \delta_2 c_0)=1 \\ k'_0 \equiv 0 \pmod{\frac{c_2}{(g_0, c_2)}}}} \frac{1}{k'_0^{1+s-w-u}} \sum_{f|k'_0} \varphi\left(\frac{k'_0}{f}\right) \sum_{n_1 \equiv 0 \pmod{\frac{k'_0}{f}}} \frac{1}{n_1^{1+s_1-s}} \\ & = \zeta(1+s_1-s) \sum_{\substack{(c_0, g_0 m'_1)=1 \\ c_0 \equiv 0 \pmod{q k_1 k_1^*}}} \frac{1}{c_0^{s+w+u}} \sum_{(f, \delta_2 c_0)=1} \frac{1}{f^{1+s-w-u}} \sum_{\substack{(\ell, \delta_2 c_0)=1 \\ \ell \equiv 0 \pmod{\frac{c_2}{(f, \frac{c_2}{(g_0, c_2)}}}}} \frac{\varphi(\ell)}{\ell^{2+s_1-w-u}}. \end{aligned}$$

As long as we assume that

$$\operatorname{Re}(1+s_1-w-u) > 0, \quad \operatorname{Re}(1+s-w-u) > 0, \quad \operatorname{Re}(s+w+u) > 0,$$

then the coprimality conditions are benign. Then we have that the Dirichlet series in (13.26) is of the form

$$\zeta(1+s_1-s) \frac{\zeta(s+w+u)}{(q k_1 k_1^*)^{s+w+u}} \zeta(1+s-w-u) \zeta(1+s_1-w-u) \left(\frac{(g_0, c_2)}{c_2}\right)^{1+\min(s, s_1)-w-u} \Delta,$$

where Δ is holomorphic and bounded by q^ε , and $\min(s, s_1)$ means the variable with the smaller real part. The factors $\zeta_q(1+s_1+s)^2 \zeta_q(1+s)$ may be absorbed into the definition of Δ provided that $\operatorname{Re}(s), \operatorname{Re}(s_1) > 0$.

Moving the summations to the inside, we derive

$$\begin{aligned} (13.27) \quad \overline{\mathcal{Q}} = & \int_{(1-2\varepsilon)} \frac{\tilde{V}(w)}{\delta_1 e_1 e_2} \int_{(1)} \frac{\gamma(\frac{1}{2} + s_1, \kappa) G(s_1)}{\gamma(\frac{1}{2}, \kappa) s_1} \int_{(1-\varepsilon)} \frac{\gamma(\frac{1}{2} + s, \kappa)^2 G(s)^2}{\gamma(\frac{1}{2}, \kappa)^2 s^2} \int_{(0)} M_2^u \tilde{\omega}(u) \\ & \frac{\zeta(1+s_1-s)}{a^{2s+s_1}} \frac{\zeta(s+w+u)}{(q k_1 k_1^*)^{s+w+u}} \zeta(1+s-w-u) \zeta(1+s_1-w-u) (q e_1 r_1)^s \left(\frac{\sqrt{q}}{e_1 r_1}\right)^{s_1} \\ & \left(\frac{(g_0, c_2)}{c_2}\right)^{1+\min(s, s_1)-w-u} \left(\frac{q}{m'_1 k_1 c_2}\right)^w \frac{\Delta \mathcal{H}(s, w, u, \kappa)}{(g_0 k_1)^{s-w-u}} \frac{(a m'_1 k_1 c_2)^{s-1/2}}{c_2^{s+w+u-1}} \frac{dudsd s_1 dw}{(2\pi i)^4}. \end{aligned}$$

For ease of reference, we list all the constraints on the variables (using $\kappa \geq 2$):

$$\begin{aligned} (13.28) \quad & 0 < \operatorname{Re}(s+w+u), \quad \operatorname{Re}(s) < 1, \\ & \operatorname{Re}(w+u) < 1 + \min(\operatorname{Re}(s), \operatorname{Re}(s_1)), \quad \operatorname{Re}(s), \operatorname{Re}(s_1) > 0. \end{aligned}$$

Now we move the contours as follows. First, move w from $1 - 2\varepsilon$ to 4ε , which does not involve crossing any poles. Following this, move s_1 to 5ε , which crosses a pole at $s_1 = s$ only. Next we move s to 6ε , which crosses a pole at $s + w + u = 1$ only. We will deal with this pole momentarily.

The pole at $s_1 = s$. This contributes to \overline{Q}

$$\frac{1}{\delta_1 e_1 e_2} \int_{(4\varepsilon)} \tilde{V}(w) \int_{(1-\varepsilon)} \frac{\gamma(\frac{1}{2} + s, \kappa)^3 G(s)^3}{\gamma(\frac{1}{2}, \kappa)^3 s^3} \int_{(0)} M_2^u \tilde{\omega}(u) \frac{\zeta(s + w + u)}{(q k_1 k_1^*)^{s+w+u}} q^{\frac{3s}{2}} \frac{\zeta(1 + s - w - u)^2}{a^{3s}} \left(\frac{g_0, c_2}{c_2} \right)^{1+s-w-u} \left(\frac{q}{m'_1 k_1 c_2} \right)^w \frac{\Delta \mathcal{H}(s, w, u, \kappa)}{(g_0 k_1)^{s-w-u}} \frac{(a m'_1 k_1 c_2)^{s-1/2}}{c_2^{s+w+u-1}} \frac{duds dw}{(2\pi i)^3}.$$

A careful scrutiny of this formula shows that it is essentially identical to (13.22) (we did not check that the Δ function is literally equal in the two cases, but this would not be surprising). Here we need that we can move w to 4ε and then u to -2ε without crossing any poles; this move in w was our first step following (13.22), so this is easily checked. Therefore, by the work in the case with all $p_i = 0$, the contribution to $\mathcal{S}_{0,0}$ from this pole is $O(q^\varepsilon)$.

The new contour. On the new line, with all the variables at multiples of ε , we have that the contribution to \overline{Q} is

$$\ll q^\varepsilon \frac{1}{\delta_1 e_1 e_2} \frac{c_2^{1/2}}{(a m'_1 k_1)^{1/2}} \frac{(g_0, c_2)}{c_2}.$$

Recalling that $\delta_1 = \frac{k_1 d}{(a, k_1 d)}$, it is not hard to see that inserting this bound into (7.10), (7.6), (7.1), gives a final contribution to $\mathcal{S}_{0,0}$ of size $O(q^\varepsilon)$.

The pole at $s = 1 - u - w$. Call this contribution to \overline{Q} by $\overline{Q}_{\text{Res}}$. Then

$$\begin{aligned} \overline{Q}_{\text{Res}} = & \int_{(4\varepsilon)} \frac{\tilde{V}(w)}{\delta_1 e_1 e_2} \int_{(5\varepsilon)} \frac{\gamma(\frac{1}{2} + s_1, \kappa) G(s_1)}{\gamma(\frac{1}{2}, \kappa) s_1} \frac{\gamma(\frac{3}{2} - u - w, \kappa)^2 G(1 - u - w)^2}{\gamma(\frac{1}{2}, \kappa)^2 (1 - u - w)^2} \\ & \int_{(0)} M_2^u \tilde{\omega}(u) \frac{\zeta(s_1 + u + w)}{(q k_1 k_1^*)} \zeta(2 - 2w - 2u) \zeta(1 + s_1 - w - u) \\ & \left(\frac{g_0, c_2}{c_2} \right)^{1+\min(1-u-w, s_1)-w-u} \left(\frac{\sqrt{q}}{e_1 r_1} \right)^{s_1} \frac{(q e_1 r_1)^{1-u-w}}{a^{s_1+2(1-u-w)}} \\ & \left(\frac{q}{m'_1 k_1 c_2} \right)^w \frac{\Delta \mathcal{H}(1 - u - w, w, u, \kappa)}{(g_0 k_1)^{1-2w-2u}} (a m'_1 k_1 c_2)^{1/2-u-w} \frac{duds_1 dw}{(2\pi i)^3}. \end{aligned}$$

The constraints $\text{Re}(w + u) < 1 + \text{Re}(s)$ and $0 < \text{Re}(s) < 1$ with $s = 1 - u - w$ simply become $0 < \text{Re}(u + w) < 1$.

Finally, we move w to $1 - 10\varepsilon$, crossing a pole at $w = s_1 - u$ only. On the new lines of integration, the contribution to \overline{Q} is

$$\ll q^\varepsilon \frac{g_0 k_1}{\delta_1 e_1 e_2} \frac{1}{k_1 k_1^*} \frac{1}{m'_1 k_1 c_2} \frac{1}{\sqrt{a m'_1 k_1 c_2}} \ll \frac{a^{1/2} q^\varepsilon}{c_2^{3/2} \sqrt{m'_1 k_1^* k_1^{3/2}}},$$

using only the weak bound $g_0 \leq \delta_1 e_1 e_2 m'_1 a$. It is easy to see that the final contribution to $\mathcal{S}_{0,0}$ from this is $O(q^\varepsilon)$.

The pole at $w = s_1 - u$. This contributes

$$\begin{aligned} \overline{\mathcal{Q}}_{\text{Res}'} := & \frac{1}{\delta_1 e_1 e_2} \int_{(5\varepsilon)} \tilde{V}(s_1 - u) \frac{\gamma(\frac{1}{2} + s_1, \kappa) G(s_1)}{\gamma(\frac{1}{2}, \kappa) s_1} \frac{\gamma(\frac{1}{2} + 1 - s_1, \kappa)^2 G(1 - s_1)^2}{\gamma(\frac{1}{2}, \kappa)^2 (1 - s_1)^2} \\ & \int_{(0)} M_2^u \tilde{\omega}(u) \frac{\zeta(2s_1) \zeta(2 - 2s_1)}{(q k_1 k_1^*)} \left(\frac{(g_0, c_2)}{c_2} \right)^{1 + \min(1 - s_1, s_1) - s_1} \left(\frac{\sqrt{q}}{e_1 r_1} \right)^{s_1} (q e_1 r_1)^{1 - s_1} \\ & \left(\frac{q}{m'_1 k_1 c_2} \right)^{s_1 - u} \frac{\Delta \mathcal{H}(1 - s_1, s_1 - u, u, \kappa)}{(g_0 k_1)^{1 - 2s_1} a^{s_1 + 2(1 - s_1)}} (a m'_1 k_1 c_2)^{1/2 - s_1} \frac{d u d s_1}{(2\pi i)^2}. \end{aligned}$$

In terms of q , this part is $O(q^\varepsilon)$, but the problem now is that the sum over m'_1 will not be absolutely convergent. The way around this roadblock is to move the contour to a location where the m'_1 -sum converges absolutely, and shift the contour back. Having $G(1/2) = 0$ once again is crucial. To this end, it is important to sum over the partition of unity in the M_1 - and M_2 -variables.

One may check that $\mathcal{H}(1 - s_1, s_1 - u, u, \kappa)$ is actually independent of u . Therefore, it is easy to sum $\overline{\mathcal{Q}}_{\text{Res}'}$ over M_2 : It is not hard to show that if $D(u)$ is a Dirichlet series absolutely convergent on the line $\text{Re}(u) = 0$, then

$$\sum_{M_2 \text{ dyadic}} \frac{1}{2\pi i} \int_{(0)} M_2^u \tilde{\omega}(u) D(u) du = D(0).$$

Now we move the s_1 -contour to $3/4$ (crossing no poles since $G(1/2) = 0$), and sum over m'_1 and M_1 , giving

$$\begin{aligned} \sum_{M_1} \sum_{m'_1} \frac{\omega_{M_1}(m'_1)}{\sqrt{m'_1}} \sum_{M_2} \overline{\mathcal{Q}}_{\text{Res}'} = & \frac{1}{\delta_1 e_1 e_2} \int_{(3/4)} \tilde{V}(s_1 - u) \frac{\gamma(\frac{1}{2} + s_1, \kappa) G(s_1)}{\gamma(\frac{1}{2}, \kappa) s_1} \\ & \frac{\gamma(\frac{3}{2} - s_1, \kappa)^2 G(1 - s_1)^2}{\gamma(\frac{1}{2}, \kappa)^2 (1 - s_1)^2} \frac{\zeta(2s_1)^2 \zeta(2 - 2s_1)}{(q k_1 k_1^*)} \left(\frac{(g_0, c_2)}{c_2} \right)^{1 + \min(1 - s_1, s_1) - s_1} \\ & \left(\frac{\sqrt{q}}{e_1 r_1} \right)^{s_1} \frac{(q e_1 r_1)^{1 - s_1}}{a^{s_1 + 2(1 - s_1)}} \left(\frac{q}{k_1 c_2} \right)^{s_1} \frac{\Delta \mathcal{H}(1 - s_1, s_1, 0, \kappa)}{(g_0 k_1)^{1 - 2s_1}} (a k_1 c_2)^{1/2 - s_1} \frac{d s_1}{2\pi i}. \end{aligned}$$

Now we are free to move the s_1 -contour back to ε , which shows that this term is bounded by

$$\frac{q^\varepsilon}{\delta_1 e_1 e_2} \frac{1}{k_1 k_1^*} \frac{(g_0, c_2)}{c_2} \frac{e_1 r_1}{g_0 k_1} \frac{1}{a^2} (a k_1 c_2)^{1/2}.$$

Using the crude bounds $\frac{(g_0, c_2)}{g_0} \leq 1$, $\frac{e_1 r_1}{\delta_1 e_1 e_2} \leq 1$, and summing trivially over k_1, d, c_2 , and a shows that this part contributes $O(q^\varepsilon)$ to $\mathcal{S}_{0,0}$.

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DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843-3368, U.S.A.

E-mail address: ekiral@tamu.edu

E-mail address: myoung@math.tamu.edu